Periodic motions of coupled impact oscillators

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Abstract We study the existence and stability of time-periodic oscillations in a chain of coupled impact oscillators, for rigid impacts without energy dissipation. We formulate the search of periodic solutions as a boundary value problem incorporating unilateral constraints. This problem is solved analytically in the vicinity of the uncoupled limit and numerically for larger coupling constants. Different solution branches corresponding to nonlinear localized modes (breathers) and normal modes are computed.

1 Introduction

Understanding the dynamics of nonlinear lattices (i.e. large networks of coupled nonlinear oscillators) is a problem of fundamental importance in mechanics, condensed matter physics or biology. One of the major issues concerns the mathematical analysis and numerical computation of special classes of nonlinear time-periodic oscillations that organize the dynamics in many situations. In particular, spatially periodic waves (standing waves or periodic traveling waves) and spatially localized waves (breathers) are the object of intensive research [16, 39]. In this context, many theoretical and numerical works have focused on smooth nonlinear systems,

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whereas relatively few mathematical existence results are available for waves in nonsmooth infinite lattices [17, 18, 26, 37]. Developing theoretical and numerical tools for the analysis of nonlinear waves in nonsmooth systems is extremely important for applications, in particular in the context of impact mechanics where unilateral contacts and friction come into play [1, 5, 8, 15, 22]. Spatially discrete lattice models are frequently encountered in this context, in particular for the modeling of waves in multibody mechanical systems (e.g. granular media) or in finite element models of continuum systems. A classical example illustrating the latter case concerns thin oscillating mechanical structures (a string under tension or a clamped beam) contacting rigid obstacles [5, 8, 22]. Such a structure can be described by a one-dimensional finite-element model involving a large number of degrees of freedom [2, 35]. The contact force between the string/beam and a rigid obstacle is either measure-valued (for rebounds with velocity jumps at contact times) or set-valued (if a wrapping of the string on the obstacle occurs), see e.g. [14].

Although nonlinear modes of oscillation have been observed in experiments on impacting mechanical systems (see e.g. [3, 8]), relatively little is known from a mathematical point of view on their existence and stability. Existence theorems for periodic and almost-periodic oscillations have been obtained in particular cases, for a continuum string model with point-mass or plane obstacle [9, 10, 11, 12, 20, 34] (see also [13] for a review). In addition, several analytical approaches have been used to obtain time-periodic solutions formally for different types of piecewise-linear dynamical systems with rigid impacts. One can mention Fourier and Green function methods [4, 5, 8, 17, 18, 19, 23, 24, 25, 31, 37], modal decomposition [27, 38] and sawtooth time transformations [32]. Most of the results obtained for discrete systems concern impacts localized on a *single particle*, and different types of waves have been constructed. In [27, 32, 38], nonsmooth normal modes have been obtained for general classes of conservative multiple degrees-of-freedom systems (the analysis of [32] is performed for a single or two impacting particles). Spatially-localized oscillations (breathers) with a single impacting node have been also studied for different classes of infinite or finite systems. Breather existence and stability has been analyzed for oscillators chains with linear nearest-neighbors coupling and a symmetric local vibroimpact potential (including in some cases a linear component), both for conservative systems [17] and forced systems with dissipative impacts [18, 31, 37].

One of the main difficulty with the above techniques is to check analytically that the formal solutions of the piecewise-linear systems are consistent, i.e. satisfy the inequality constraints corresponding to non-penetration of the obstacles. This has been achieved in a number of works in the case of breathers [17, 18, 37] and for nonsmooth modes close to grazing linear normal modes [27]. In the work [19], the analysis of [31] has been extended to several impacting particles, but the verification of the inequality constraints is still an open problem in that case.

In this work, we study the existence and stability of time-periodic oscillations in an infinite chain of linearly coupled impact oscillators reminiscent of a model analyzed in [19, 31], for rigid impacts without energy dissipation. We show the existence of exact solutions (i.e. check the non-penetration conditions) for an arbitrary number of impacting particles when the coupling between oscillators is small, and we compute solution branches numerically for larger couplings. The system under consideration is depicted in Fig. 1. Particle positions are denoted as $y(t) = (y_n(t))_{n\in\mathbb{Z}}$ and satisfy the following complementarity system

$$\ddot{y}_n + y_n - \gamma(\Delta y)_n = \lambda_n, \quad n \in \mathbb{Z}, \tag{1}$$

$$0 \le \lambda \perp (y + \mathbb{1}) \ge 0, \tag{2}$$

$$\text{if } \dot{y}_n(t^-) < 0 \text{ and } y_n(t) = -1 \text{ then } \dot{y}_n(t^+) = -\dot{y}_n(t^-), \tag{3}$$

where $(\Delta y)_n = y_{n+1} - 2y_n + y_{n-1}$ defines a discrete Laplacian operator, $\mathbb{1}$ denotes the constant sequence with all terms equal to unity and $\gamma \ge 0$ is a parameter. Nondissipative impacts occur for $y_n(t) = -1$ and give rise to impulsive reaction forces $\lambda_n(t)$. This configuration differs from the case of a symmetric local vibroimpact potential considered in [19, 31] which introduces an additional barrier above the chain.

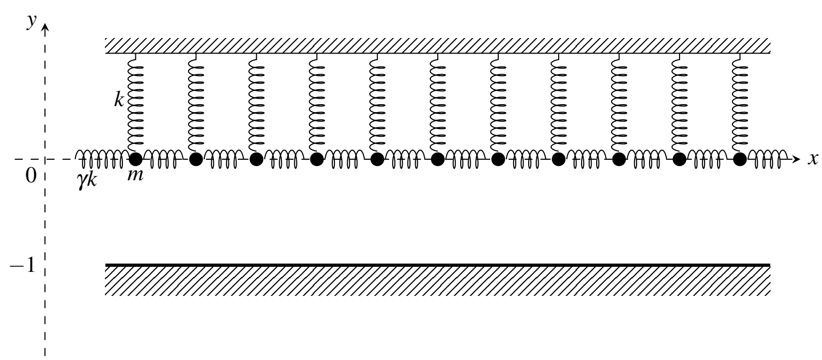


Fig. 1: A chain of identical impact oscillators with linear nearest-neighbors coupling. The chain is allowed to oscillate above a straight obstacle. After suitable rescaling, the obstacle position is fixed to $y = -1$, and the masses m of particles and local stiffness k are set to unity.

Our analytical results are presented in section 2. We start by describing in section 2.1 some simple examples of nonsmooth modes of oscillations (in-phase, out-of-phase, and some symmetry-breaking bifurcations from these modes). In section 2.2, we reformulate the search of periodic solutions of (1)-(2)-(3) as a boundary value problem incorporating unilateral constraints. This formulation, together with an appropriate notion of nondegenerate modes introduced in section 2.3, allows us to construct nonsmooth modes of oscillations (spatially localized or extended) at small coupling (see theorems 1 and 2). This approach is an adaptation of the idea of "anticontinuum" limit [16, 29, 36] to the nonsmooth setting. Section 2.4 deals with the linear stability of time-periodic solutions of (1)-(2)-(3). We provide a formula for the monodromy matrix that determines spectral stability in the presence of simple impacts, following the lines of [42]. In section 3, the above results are used for the numerical computation of time-periodic solutions. Solution branches are continued for fixed values of T, varying the linear stiffness γ (and starting from the limit $\gamma = 0$) or by fixing γ and varying T. We compute in this way some families of breathers and extended modes and study their linear stability. Dynamical instabilities are illustrated by integrating (1)-(2)-(3) numerically. These computations are performed with the Siconos software for nonsmooth dynamical systems [1, 43].

2 Analytical study of nonsmooth modes

2.1 Definitions and basic examples

We look for T-periodic solutions of (1)-(2)-(3) even in time, and assume each particle undergoes at most one impact during each period of oscillation. Consequently, for a given particle, impacts either occur at half-period multiples or do not occur at all. We denote by $I_k \subset \mathbb{Z}$ with $k = 1$ or 2 the index sets of particles impacting at $t = (2m+k)\,T/2$ for all $m \in \mathbb{Z}$ (i.e. $y_n((2m+k)\,T/2) = -1$), and by $I_0 := \mathbb{Z} \setminus (I_1 \cup I_2)$ the index set corresponding to non-impacting particles (i.e. $y_n(t) > -1$ for all t). We have thus $\lambda_n = 0$ for all $n \in I_0$ and

$$\lambda_n = 2\,\dot{y}_n\Big(\frac{kT^+}{2}\Big) \sum_{m\in\mathbb{Z}} \delta_{(m+\frac{k}{2})T} \quad \text{for all } n \in I_k. \tag{4}$$

The triplet (I_0, I_1, I_2) will be denoted as the *pattern* of the periodic solution. A *nonsmooth mode* corresponds to a continuous one-parameter family of periodic solutions (parameterized typically by T) sharing a given pattern with $I_0 \neq \mathbb{Z}$ (i.e. impacts occur).

We provide below some simple examples of nonsmooth modes. The simplest case corresponds to the in-phase mode with $I_1 = \mathbb{Z}$ (or equivalently $I_2 = \mathbb{Z}$ up to a phase shift). This solution exists for $T \in (\pi, 2\pi)$ and reads

$$y_n(t) = -\frac{\cos t}{\cos(T/2)} \text{ for } |t| \leq T/2, \tag{5}$$

where (5) is extended by periodicity outside the interval $(-T/2, T/2)$. The impact velocity reads in particular $\dot{y}_1((T/2)^+) = -\dot{y}_1((T/2)^-) = -\tan(T/2)$. The amplitude of oscillations diverges when $T \to \pi$ and becomes unity for $T = 2\pi$. In that case, the impact becomes grazing (i.e. occurs at zero velocity) and one recovers the *linear* in-phase mode $y_n(t) = \cos t$, which is solution of (1) with $\lambda = 0$. Notice that for $T \neq 2k\pi$ outside the interval $(\pi, 2\pi)$, expression (5) does not provide a solution of (1)-(2)-(3) because the constraint $y_n \geq -1$ is violated.

Another example concerns nonsmooth modes with spatial period two, i.e. which satisfy $y_{n+2}(t) = y_n(t)$. Nonsmooth modes in two degrees-of-freedom impacting systems have been studied in a number of works, see e.g. [30, 40] in the case of symmetric constraints and [22] for more references. In what follows we discuss the case when I_1 and I_2 consist of the sets of odd and even integers respectively. Moreover we assume that all impacts velocities are identical and nonzero. In order to compute such modes, we introduce the relative displacement $r = y_2 - y_1$, the center of mass $q = (y_1 + y_2)/2$ and the impact velocity $v = \dot{y}_2(0^+) = \dot{y}_1((T/2)^+) \neq 0$. From equations (1) and (4) taken at $n = 1,2$ and considering the spatial period two of the mode, one obtains

$$\ddot{r} + \Omega^2 r = 2v \sum_{m \in \mathbb{Z}} (-1)^m \delta_{m\frac{T}{2}}, \tag{6}$$

where $\Omega = \sqrt{1+4\gamma}$. Note that Ω is the frequency of the linear out-of-phase mode $y_n(t) = (-1)^n \cos(\Omega t)$, which is solution of (1) with $\lambda = 0$. If the non-resonance condition $(2m+1)(2\pi/T) \neq \Omega$ holds true for all integers m, there exists an even T-periodic solution of (6) defined by

$$r(t) = \frac{v}{\Omega} \frac{\sin(\Omega(t - \frac{T}{4}))}{\cos(\Omega T/4)} \text{ for } t \in [0, T/2], \tag{7}$$

where the integration constants have been determined from the conditions $v = \dot{r}(0^+) = \dot{r}((T/2)^-)$. In addition, the T-periodic solution is unique if $m(2\pi/T) \neq \Omega$ for all integers m. From expression (7) and using the fact that r is T-periodic and even, one can notice that $r(\frac{T}{4} + t) = -r(\frac{T}{4} - t)$ for all $t \in \mathbb{R}$.

Similarly, the center of mass satisfies

$$\ddot{q} + q = v \sum_{m \in \mathbb{Z}} \delta_{m\frac{T}{2}}. \tag{8}$$

Let us assume that the non-resonance condition $T \neq 4m\pi$ holds true for all integer m. In that case, equation (8) admits an even $T/2$-periodic solution. Indeed, since $v/2 = \dot{q}(0^+) = -\dot{q}((T/2)^-)$, we find

$$q(t) = \frac{v}{2} \frac{\cos(t - \frac{T}{4})}{\sin(T/4)} \text{ for } t \in [0, T/2], \tag{9}$$

and q is defined as the $T/2$-periodic extension of (9). The symmetry $q(\frac{T}{4} + t) = q(\frac{T}{4} - t)$ for all $t \in \mathbb{R}$ and the fact that q is $T/2$-periodic imply that q is even. In addition, (8) does not possess additional T-periodic solutions if the non-resonance condition $T \neq 2m\pi$ holds true for all integer m.

Particle displacements are obtained from the identities

$$y_1 = q - \frac{r}{2}, \quad y_2 = q + \frac{r}{2}.$$

One can check that $\dot{y}_1(0^+) = 0$, hence $\dot{y}_1(0^-) = 0$ and y_1 is smooth everywhere except at the impact times $t = (2k+1)T/2$ with $k \in \mathbb{Z}$. Moreover, it follows from the symmetries of r that $y_2(t) = y_1(t+T/2) = y_1(t-T/2)$.

We use the constraint $y_2(0) = -1$ to determine v from T, which yields

$$v = 2\left(\frac{1}{\Omega}\tan(\Omega T/4) - \cot(T/4)\right)^{-1} \tag{10}$$

and implies $y_1(T/2) = -1$. The expression in (10) is depicted in Figure 2. In the uncoupled case $\gamma = 0$, expression (10) simplifies to $v = -\tan(T/2)$ and one recovers the case $n = 1$ of (5). Moreover, in the limit cases $T \to (2k+1)\,2\pi/\Omega$ ($k \in \mathbb{N}_0$) and $T \to 4m\pi$ ($m \in \mathbb{N}$) one obtains $v \to 0$, i.e. a grazing impact. When $T \to (2k+1)\,2\pi/\Omega$ and $\Omega \neq (2k+1)/(2m)$ for all $m \in \mathbb{N}$, the above solution converges towards the linear out-of-phase mode $y_n(t) = (-1)^{n+1}\cos(\Omega t)$, while $T \to 4m\pi$ and $\Omega \neq (2k+1)/(2m)$ for all $k \in \mathbb{N}_0$ leads to a convergence towards the linear in-phase mode $y_n(t) = -\cos t$.

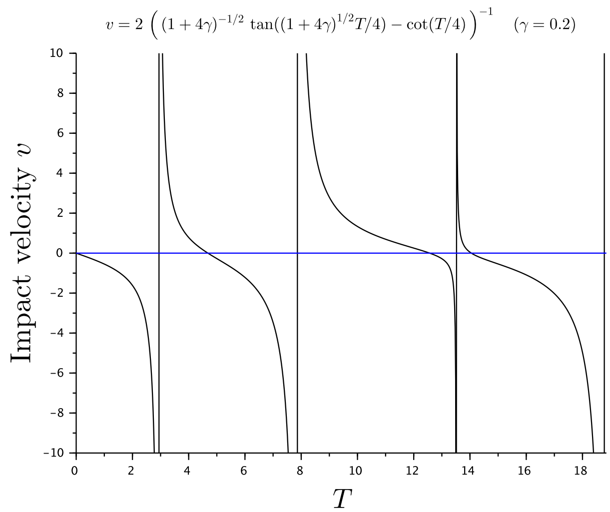


Fig. 2: Impact velocity as a function of the period

In order to obtain solutions of (1)-(2)-(3), there remains to check for which values of parameters γ, T the constraint $y_1 \geq -1$ is satisfied. Let us examine this problem when the coupling constant γ is fixed and T is varied. A necessary condition is $v \geq 0$, which is achieved for values of $T > 0$ within an infinite and unbounded sequence of disjoint intervals depending of γ. The lower bounds of these intervals are the roots of v^{-1}, and the upper bounds take the form $T = (2k+1)\,2\pi/\Omega$ with $k \in \mathbb{N}_0$ or $T = 4m\pi$ with $m \in \mathbb{N}$ (values leading to $v = 0$). In particular, the first interval takes the form $(T_0(\gamma), 2\pi/\Omega]$, where $T_0(\gamma)$ is implicitely defined through

$$\frac{1}{\Omega}\tan(\Omega\, T_0/4) = \cot(T_0/4), \quad T_0 \in (0, 2\pi/\Omega). \tag{11}$$

Note that $\lim_{\gamma\to+\infty} T_0(\gamma) = 0$ (since $T_0 < 2\pi\,(1+4\gamma)^{-1/2}$), $\lim_{\gamma\to 0} T_0(\gamma) = \pi$ (case $\Omega = 1$ of (11)), and T_0 is a decreasing function of γ (since the left side of (11) is increasing with Ω or γ), hence $T_0(\gamma) < \pi$ for $\gamma > 0$. The upper bound $T = 2\pi/\Omega$ yields $v = 0$ (grazing impact) as previously outlined, whereas in the case $T \to T_0(\gamma)^+$ one obtains $v \to +\infty$.

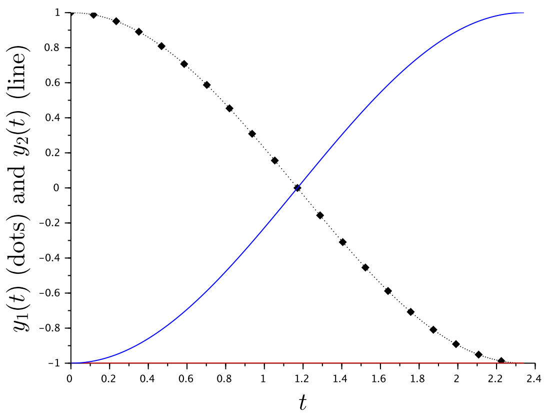
Now let us check the constraint $y_1(t) \geq -1$ in the case $T \in (T_0(\gamma), 2\pi/\Omega)$. One can restrict the discussion to $t \in [0, T/2]$ without loss of generality (since y_1 is even and T-periodic). In that case, we deduce from the above computations

$$\dot{y}_1(t) = -\frac{v}{2}\left(\frac{\sin(t-\frac{T}{4})}{\sin(T/4)} + \frac{\cos(\Omega\,(t-\frac{T}{4}))}{\cos(\Omega\, T/4)}\right).$$

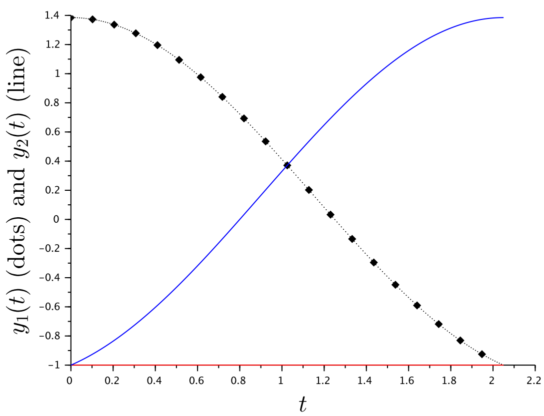
Consequently, the conditions $T < 2\pi/\Omega < 2\pi$ and $v > 0$ (which follows from $T \in (T_0(\gamma), 2\pi/\Omega))$ imply that y_1 is decreasing on $[T/4, T/2]$, hence $y_1(t) > -1 = y_1(T/2)$ for all $t \in [T/4, T/2)$. In addition, expressions (7) and (9) show that $r \leq 0$ and $q > 0$ on $[0, T/4]$, hence $y_1 > 0$ on $[0, T/4]$. This shows that $y_1(t) > -1$ for all $t \in [0, T/2)$.

As a conclusion, we have obtained a family of even and time-periodic solutions of (1)-(2)-(3) parameterized by their period $T \in (T_0(\gamma), 2\pi/\Omega)$. These solutions have spatial period two and possess the symmetry $y_{n+1}(t) = y_n(t + T/2)$. When $T \to T_0(\gamma)^+$, the impact velocity v and amplitude of oscillations $y_1(0)$ diverge. When $T \to 2\pi/\Omega$, the mode converges towards the linear out-of-phase mode. This family of solutions will be denoted as *nonsmooth out-of-phase mode*. They are illustrated for several values of T in Figure 3.

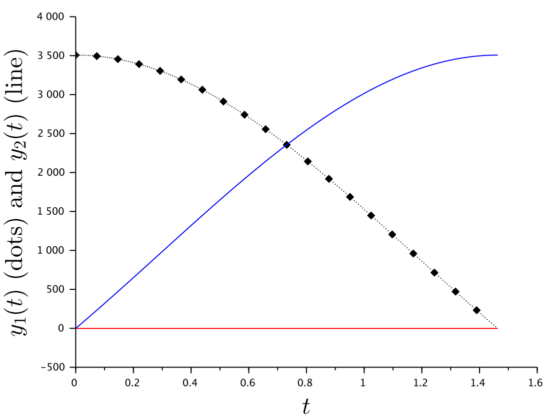
There exist other nonsmooth modes with spatial period 2 and $I_0 = \varnothing$, $I_2 = 2\mathbb{Z}$ not discussed above, for example a branch of solutions emerging above $T = 4\pi/\Omega$. For $T = 4\pi/\Omega$, odd particles undergo a grazing impact at $t = 0$ (we conjecture the existence of a nonsmooth mode with two impacts per period and $T < 4\pi/\Omega$). When T increases above $4\pi/\Omega$, no impacts occur at $t = 0$ for odd particles and the branch of solutions can evolve in different ways depending on γ. If $\gamma < 5/16$ (so that $4\pi < 6\pi/\Omega$), the mode converges towards the linear in-phase mode when $T \to 4\pi^-$ (this corresponds to a period-doubling bifurcation of the in-phase mode), a limit in which odd particles again display a grazing impact at $t = 0$. If $\gamma > 5/16$ (case $6\pi/\Omega < 4\pi$), convergence towards the linear out-of-phase mode takes place when $T \to (6\pi/\Omega)^-$ (period-tripling bifurcation of the out-of-phase mode). In this limit, odd particles undergo a grazing impact at $t = \pi/\Omega$. Illustrations of period doubling bifurcations are displayed in Figure 4 and period tripling bifurcations in Figure 5.



(a) Particle oscillations for $\gamma = 0.2$, $\quad T = 2\pi(1+4\gamma)^{-1/2} \approx 4.68$

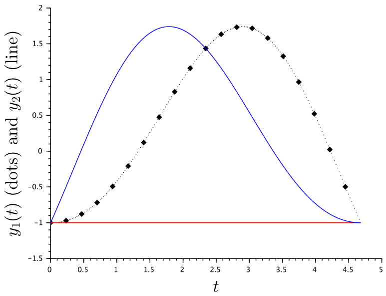


(b) Particle oscillations for $\gamma = 0.2$, $\quad T = 4.1$

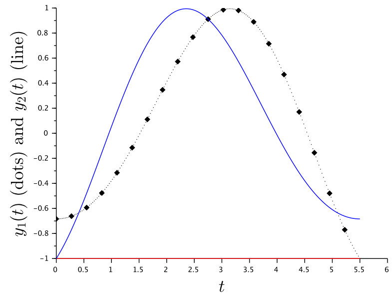


(c) Particle oscillations for $\gamma = 0.2$, $\quad T = 2.926$

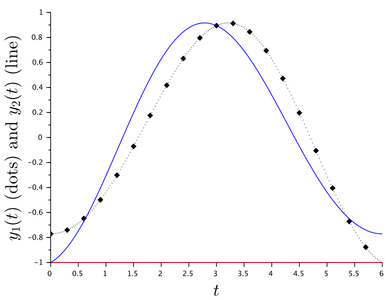
Fig. 3: Nonsmooth out-of-phase modes for several values of T



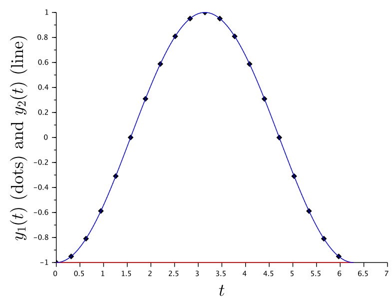
(a) Particle oscillations for $\gamma = 0.2, T = 4\pi(1+4\gamma)^{-1/2}$



(b) Particle oscillations for $\gamma = 0.2, T = 11$



(c) Particle oscillations for $\gamma = 0.2, T = 12$



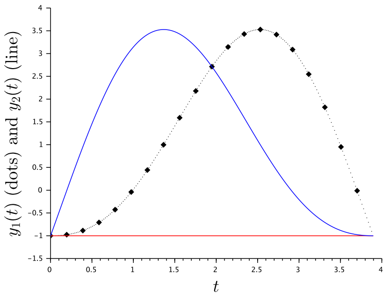
(d) Particle oscillations for $\gamma = 0.2, T = 4\pi$

Fig. 4: Period doubling bifurcation

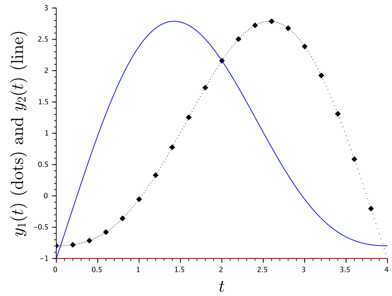
2.2 Boundary value problem

In the sequel, E denotes either the Banach space $\ell_\infty(\mathbb{Z})$ of real bounded sequences on $\mathbb{Z}$, the Hilbert space $\ell_2(\mathbb{Z})$ of square-summable sequences, or the Hilbert space $\mathscr{P}^p$ of p-periodic sequences (isomorphic to the Euclidean space $\mathbb{R}^p$) for a fixed integer p. The case $E = \ell_2(\mathbb{Z})$ will be relevant for the study of localized modes, and the periodic case will be considered for numerical computations. We consider a chain of impact ocillators with positions described by a vector $y(t) \in E$ solution of the complementarity system (1)-(2)-(3). We look for T-periodic solutions even in time, with a prescribed pattern (I_0, I_1, I_2) (as defined in section 2.1) such that $I_0 \neq \mathbb{Z}$.

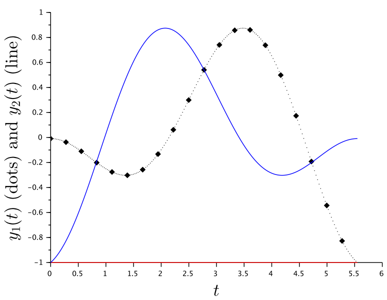
The splitting $\mathbb{Z} = I_0 \cup I_1 \cup I_2$ allows one to identify E with $E^{(0)} \times E^{(1)} \times E^{(2)}$ where $E^{(k)}$ is a space of sequences indexed by $n \in I_k$, equiped with the same norm as E ($\| \|_2$ or $\| \|_\infty$). For all $y \in E$, we shall use the notation $y = (y^{(0)}, y^{(1)}, y^{(2)})$ with $y^{(k)} = (y_n)_{n \in I_k} \in E^{(k)}$. Any solution of the linear differential equation (12) satisfies $\dot{y}(t) \in E$, therefore we shall denote $\dot{y} = (\dot{y}^{(0)}, \dot{y}^{(1)}, \dot{y}^{(2)})$ with $\dot{y}^{(k)} \in E^{(k)}$. The above



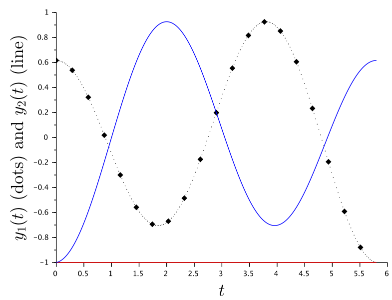
(a) Particle oscillations for $\gamma = 0.4, T = 4\pi(1+4\gamma)^{-1/2}$



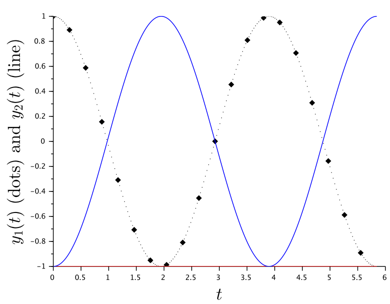
(b) Particle oscillations for $\gamma = 0.4, T = 8$



(c) Particle oscillations for $\gamma = 0.4, T = 11.1$



(d) Particle oscillations for $\gamma = 0.4, T = 11.6$



(e) Particle oscillations for $\gamma = 0.4, T = 6\pi(1+4\gamma)^{-1/2}$

Fig. 5: Period tripling bifurcation

problem can be reformulated as a boundary value problem on a half-period interval $(0,T/2)$,

$$\ddot{y}_n + y_n - \gamma(\Delta y)_n = 0, \quad n \in \mathbb{Z}, \quad t \in (0,T/2), \tag{12}$$

with boundary conditions

$$\begin{aligned} \dot{y}^{(i)}(0) = 0 \text{ for } i \in I_0 \cup I_1 \ , \ y^{(2)}(0) = -\mathbb{1}, \\ \dot{y}^{(i)}(T/2) = 0 \text{ for } i \in I_0 \cup I_2 \ , \ y^{(1)}(T/2) = -\mathbb{1}, \end{aligned} \tag{13}$$

and constraint

$$y(t) + \mathbb{1} > 0, \quad t \in (0,T/2). \tag{14}$$

Indeed, it is immediate that any even T-periodic solution of (1)-(2)-(3) with pattern (I_0,I_1,I_2) satisfies (12)-(13)-(14). Moreover, every solution of (12)-(13)-(14) can be extended to an even T-periodic function y, which in turn defines a solution of (1)-(2)-(3). Indeed, since $\dot{y}$ is odd, we have $\dot{y}(0^-) = -\dot{y}(0^+)$ and thus $\dot{y}((kT)^-) = -\dot{y}((kT)^+)$ for all $k \in \mathbb{Z}$ because $\dot{y}$ is T-periodic. In the same way, since $\dot{y}$ is odd and T-periodic, we have $\dot{y}((T/2)^-) = -\dot{y}((-T/2)^+) = -\dot{y}((T/2)^+)$, and thus we have by periodicity $\dot{y}(((2k+1)T/2)^-) = -\dot{y}(((2k+1)T/2)^+)$ for all $k \in \mathbb{Z}$.

In what follows, we reformulate the boundary value problem (12)-(13) as a linear system for $\xi = (y^{(0)}(0), y^{(1)}(0), \dot{y}^{(2)}(0)) \in E^{(0)} \times E^{(1)} \times E^{(2)}$, i.e. as an affine equation in E. For this purpose, we define the projection $P : E \times E \to E$ through

$$P(y,\dot{y}) = (\dot{y}^{(0)}, y^{(1)}, \dot{y}^{(2)})$$

and an embedding $N : E \to E \times E$ by

$$N(y^{(0)}, y^{(1)}, \dot{y}^{(2)}) = (u,v), \quad u = (y^{(0)}, y^{(1)}, 0), \ v = (0,0,\dot{y}^{(2)}) \text{ in } E^{(0)} \times E^{(1)} \times E^{(2)}.$$

Introducing $Y = (y,\dot{y})^T \in E \times E$, the linear differential equation (12) takes the form

$$\dot{Y} = JY + \gamma LY, \tag{15}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ \Delta & 0 \end{pmatrix}$$

and $\mathbb{I}$ is the identity map in E. Let us denote by $S_\gamma(t) = e^{(J+\gamma L)t} \in \mathscr{L}(E \times E)$ the flow of (15).

The boundary condition at $t = 0$ defined in (13) takes the form $Y(0) = N\xi - B$, where $B = (\mathbb{1}_{I_2}, 0)^T \in E \times E$ and $\mathbb{1}_{I_2}$ denotes the indicator function of I_2. Moreover, the boundary condition at $t = T/2$ in (13) reads $PY(T/2) = -\mathbb{1}_{I_1}$. Consequently, the boundary value problem (12)-(13) is equivalent to

$$M_{\gamma,T}\,\xi = \eta, \tag{16}$$

where $M_{\gamma,T} = P S_\gamma(T/2) N \in \mathscr{L}(E)$ and $\eta = P S_\gamma(T/2) B - \mathbb{1}_{I_1}$.

In the case $E = \mathscr{P}^p$ (periodic boundary conditions with period p), E is isomorphic to $\mathbb{R}^p$ and (16) takes the form of a p-dimensional linear system. The solution $\xi \in E$ can be identified with a vector $x \in \mathbb{R}^p$ defined by

$$x_i = y_i \;\text{ if }\; i \in I_0 \cup I_1, \quad x_i = \dot{y}_i \;\text{ if }\; i \in I_2.$$

The matrix $P \in M_{p,2p}(\mathbb{R})$ reads

$$P_{j,j} = 1 \;\text{ if }\; j \in I_1, \quad P_{j,j+p} = 1 \;\text{ if }\; j \in I_0 \cup I_2, \quad P_{i,j} = 0 \;\text{ elsewhere.}$$

The matrix $N \in M_{2p,p}(\mathbb{R})$ is defined by

$$N_{i,i} = 1 \;\text{ if }\; i \in I_0 \cup I_1, \quad N_{i+p,i} = 1 \;\text{ if }\; i \in I_2, \quad N_{i,j} = 0 \;\text{ elsewhere.}$$

2.3 Nondegenerate modes and continuation at small coupling

Consider an even T-periodic solution of (1)-(2)-(3) with pattern (I_0, I_1, I_2) (recall that under these assumptions, each particle undergoes at most one impact per period). The reduced initial condition $\xi = (y^{(0)}(0), y^{(1)}(0), \dot{y}^{(2)}(0)) \in E^{(0)} \times E^{(1)} \times E^{(2)}$ defines a solution of the linear problem (16). This leads us to introduce the following notion of *nondegenerate* periodic solution.

Definition 1. An even T-periodic solution of (1)-(2)-(3) with pattern (I_0, I_1, I_2) is nondegenerate if the map $M_{\gamma,T}$ is invertible and

$$\dot{y}_n((T/2)^-) < 0 \;\forall n \in I_1, \quad \dot{y}_n(0^+) > 0 \;\forall n \in I_2. \tag{17}$$

Let us consider any nondegenerate periodic solution of (1)-(2)-(3). Since $M_{\gamma,T}$ depends analytically on γ, T, the corresponding solution of (16) admits *locally* a unique continuation with respect to (γ, T) denoted by $\xi_{\gamma,T}$, which is analytic in (γ, T) in some open set [41]. It follows that

$$Y_{\gamma,T}(t) = (y_{\gamma,T}(t), \dot{y}_{\gamma,T}(t))^T = S_\gamma(t)\,(N\,\xi_{\gamma,T} - B) \tag{18}$$

is a solution of (12) satisfying (13).

In order to check the constraint (14), we define $u_{\gamma,T}(t) = y_{\gamma,T}(\frac{T}{2}\,t) + \mathbb{1}$ and introduce the Banach space

$$X = \{\, u \in C^1([0,1], E), \quad u_n(1) = 0 \;\forall n \in I_1, \quad u_n(0) = 0 \;\forall n \in I_2 \,\},$$

equiped with the C^1-norm. We consider the *open* set

$$\begin{aligned} \Omega = \{\, & u \in X,\ \forall n \in I_0,\ u_n > 0 \text{ on } [0,1], \\ & \forall n \in I_1,\ u_n > 0 \text{ on } [0,1),\ \dot{u}_n(1^-) < 0, \\ & \forall n \in I_2,\ u_n > 0 \text{ on } (0,1],\ \dot{u}_n(0^+) > 0 \,\}. \end{aligned}$$

Thanks to assumption (17), the nondegenerate periodic solution belongs to Ω. Since the map $(\gamma,T) \mapsto u_{\gamma,T}$ is continuous in X, the local continuation with respect to (γ,T) of the nondegenerate solution stays locally in Ω, and thus the constraint (14) is satisfied by $y_{\gamma,T}$ when (γ,T) lies in some open set $\mathscr{U}$. Consequently, we have obtained a family of solutions of the boundary value problem (12)-(13)-(14) parameterized by (γ,T), which provides in turn a family of solutions of (1)-(2)-(3). As a conclusion, we have shown the following.

Theorem 1. *Any nondegenerate even periodic solution of (1)-(2)-(3) with a given pattern persists for values of the coupling constant γ and period T lying in an open set $\mathscr{U}$. Moreover, these solutions take the form $y(t) = y_{\gamma,T}(t)$ for all $t \in [0,T/2]$, where the map $(t,\gamma,T) \mapsto y_{\gamma,T}(t)$ is analytic in $\mathbb{R}\times\mathscr{U}$ and defined in (18).*

In particular, the above result shows that any nondegenerate periodic solution is part of a continuous branch of periodic solutions parameterized by T and forming a nonsmooth mode. The continuation may stop when a new grazing impact takes place for $n \in I_0$ or if an impact occurring for $n \in I_1$ or I_2 becomes grazing. In such cases, the branch of periodic solutions might be continued with a different pattern or by allowing several impacts per period or sticking contacts, but these extensions are out of the scope of the present study.

Another case when the above continuation theorem does not apply corresponds to the noninvertibility of $M_{\gamma,T}$. This situation may lead to a divergence of the solution (i.e. divergence of $\|(y^{(0)}(0), y^{(1)}(0), \dot{y}^{(2)}(0))\|$) or to a bifurcation of periodic solutions.

The solution of (12)-(13) is non-unique, or equivalently $M_{\gamma,T}$ admits a nontrivial kernel, if and only if the homogeneous boundary value problem given by (12) and

$$\begin{aligned} \dot{y}^{(i)}(0) = 0 \text{ for } i \in I_0 \cup I_1\ ,\ y^{(2)}(0) = 0, \\ \dot{y}^{(i)}(T/2) = 0 \text{ for } i \in I_0 \cup I_2\ ,\ y^{(1)}(T/2) = 0, \end{aligned} \tag{19}$$

admits nontrivial solutions $y(t) \in E$. Let us fix $E = \ell_\infty(\mathbb{Z})$ and discuss some *resonant* cases when this phenomenon occurs. The linear equation (12) admits normal mode solutions (or "phonons")

$$y_n(t) = a \cos(\Omega_q t + \varphi) \cos(qn + \psi), \tag{20}$$

whose frequencies $\Omega_q = (1+4\gamma \sin^2(q/2))^{1/2}$ span the phonon band $[1,\Omega]$, the highest frequency $\Omega = \sqrt{1+4\gamma}$ corresponding to the out-of-phase mode with $q = \pi$. For nonsmooth modes having certain patterns, simple nontrivial solutions of (12)-(19) can be found in the form (20) if some multiple of π/T belongs to the phonon band.

For example, if $I_1 = \mathbb{Z}$ or $I_2 = \mathbb{Z}$ (this is the case for the in-phase mode) and if one has a resonance $(2m+1)\pi/T = \Omega_q$ for some integer m and $q \in [0,\pi]$, then (20) provides nontrivial solutions of (12)-(19), and thus $M_{\gamma,T}$ is non-invertible. This occurs e.g. for $T = \pi$ ($m = 0$, $q = 0$), where the amplitude of the in-phase mode becomes infinite.

Moreover, if one considers a localized pattern $I_0 = \mathbb{Z}\setminus\{n_0\}$ for some integer n_0, then the resonance $m\,(2\pi/T) = \Omega_q$ ($m \in \mathbb{N}$) leads to nontrivial solutions of (12)-(19) (obtained by choosing $\psi = \frac{\pi}{2} - q\,n_0$ in (20)) and thus $M_{\gamma,T}$ is non-invertible.

In the case $E = \mathscr{P}^p$ (p-periodic sequences), the phonon band becomes discrete (wavenumbers take the form $q = k\,2\pi/p$ with $k \in \mathbb{Z}$), but the above resonance conditions remain valid when $I_1 = \mathbb{Z}$ or $I_2 = \mathbb{Z}$, or if $I_0 = \mathbb{Z}\setminus\{n_0 + p\,\mathbb{Z}\}$.

As an application of theorem 1, we now prove the existence of nonsmooth modes having any type of pattern, close to the uncoupled (or "anticontinuum") limit $\gamma = 0$. In theorem 2 below, the mode pattern $I = (I_0, I_1, I_2)$ must be compatible with the choice of E. For $E = \mathscr{P}^p$, the sets I_k are assumed invariant modulo p, and for $E = \ell_2(\mathbb{Z})$ the sets I_1 and I_2 have to be finite (no impacts occur at infinity when oscillations are spatially localized). In the case $E = \ell_\infty(\mathbb{Z})$ there are no restrictions on the mode pattern.

Theorem 2. *Fix a mode pattern $I = (I_0, I_1, I_2)$ compatible with E. There exists an open set $\mathscr{V} \subset \mathbb{R}^2$ including the segment $\{0\}\times(\pi, 2\pi)$, such that for all $(\gamma, T) \in \mathscr{V}$, system (1)-(2)-(3) admits a unique even periodic solution with pattern I, which is defined by (18).*

Proof. It suffices to check that for $\gamma = 0$ and all $T \in (\pi, 2\pi)$, system (1)-(2)-(3) admits a unique nondegenerate periodic solution with pattern I. Then the result follows by direct application of theorem 1.

Let us denote by $y_n^{\mathrm{ip}}(t)$ the in-phase mode defined by (5) with period $T \in (\pi, 2\pi)$. For $\gamma = 0$, system (1)-(2)-(3) consists of uncoupled impact oscillators. Consequently, the unique T-periodic solution with pattern I is given by $y_n = y_n^{\mathrm{ip}}$ for all $n \in I_1$, $y_n(t) = y_n^{\mathrm{ip}}(t + T/2)$ for all $n \in I_2$, and $y_n = 0$ for all $n \in I_0$ (for $\gamma = 0$, all non-impacting nontrivial solutions are 2π-periodic, and we have assumed $T < 2\pi$). It follows that the condition (17) of non-grazing impacts is satistied for $T \in (\pi, 2\pi)$. In order to show that the T-periodic solution obtained for $\gamma = 0$ is nondegenerate, there remains to check that the linear map $M_{0,T}$ of (16) is invertible. We have for all $\xi = (\xi^{(0)}, \xi^{(1)}, \xi^{(2)}) \in E^{(0)} \times E^{(1)} \times E^{(2)}$

$$M_{0,T}\,\xi = P\,e^{JT/2}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{21}$$

where $u, v \in E = E^{(0)} \times E^{(1)} \times E^{(2)}$ are defined as follows

$$u = (\xi^{(0)}, \xi^{(1)}, 0), \quad v = (0, 0, \xi^{(2)}).$$

Moreover, we have in the block form

$$e^{Jt} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathscr{L}(E \times E),$$

hence (21) yields

$$M_{0,T}\,\xi = P\,(y, \dot{y}),$$

where $y, \dot{y} \in E = E^{(0)} \times E^{(1)} \times E^{(2)}$ are defined by

$$y = (\cos(T/2)\,\xi^{(0)}, \cos(T/2)\,\xi^{(1)}, \sin(T/2)\,\xi^{(2)}),$$

$$\dot{y} = (-\sin(T/2)\,\xi^{(0)}, -\sin(T/2)\,\xi^{(1)}, \cos(T/2)\,\xi^{(2)}).$$

Consequently, $M_{0,T} \in \mathscr{L}(E^{(0)} \times E^{(1)} \times E^{(2)})$ takes the following diagonal form

$$M_{0,T}\,\xi = (-\sin(T/2)\,\xi^{(0)}, \cos(T/2)\,\xi^{(1)}, \cos(T/2)\,\xi^{(2)}).$$

It follows that $M_{0,T}$ is invertible because the coefficients $\cos(T/2)$ and $\sin(T/2)$ do not vanish for $T \in (\pi, 2\pi)$. □

It is interesting to compare the local continuation result of theorem 2 and the explicit computations of the nonsmooth in-phase and out-of-phase modes performed in section 2. The in-phase mode actually exists for all $\gamma \in \mathbb{R}$ and $T \in (\pi, 2\pi)$. Moreover, the out-of-phase mode exists for all $\gamma \geq 0$ (and even for γ slightly negative) and $T \in (T_0(\gamma), 2\pi(1+4\gamma)^{-1/2})$.

2.4 Stability

In this section, the linear stability of periodic solutions is analyzed through the eigenvalues of an associated monodromy matrix. Since the trajectory of the state of the system is nonsmooth at impact times, some precautions must be taken into account to compute the monodromy matrix. The computation of the monodromy follows the line of the work in [42].

In this section, we will consider the finite-dimensional case $E = \mathscr{P}^p$. For a given initial condition $Y_0 = (y(t_0), \dot{y}(t_0))^T \in \mathbb{R}^{2p}$, the conservative system (1)-(2)-(3) admits a unique solution (without accumulation of impacts) which is analytic in time between impacts [6, 7, 33]. Let us define the trajectory of the flow of (1)-(2)-(3) for the initial conditions (t_0, Y_0) as

$$\begin{array}{rcl} \phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2p} & \to & \mathbb{R}^{2p} \\ (t, t_0, Y_0) & \mapsto & \phi(t, t_0, Y_0). \end{array} \tag{22}$$

The flow ϕ satisfies $\phi(t_0, t_0, Y_0) = Y_0$. The trajectory of the system for the initial condition (t_0, Y_0) is $Y(t) = \phi(t, t_0, Y_0)$. In the sequel we consider a time t and an initial time t_0 at which no impact occurs. The computation of the monodromy amounts to performing the differentiation of the flow ϕ at time t for the initial time t_0 with respect to the initial condition Y_0, that is

$$M(t) = \frac{d\phi(t, t_0, Y_0)}{dY_0}. \tag{23}$$

This matrix can be approximated by finite differences. As it is noted in [42], the application of a finite-difference scheme may result in a poor approximation of the monodromy matrix. Since, in our application, the flow can be defined as a concatenation of piecewise smooth flows between impact times, we present here a closed-form formula for the monodromy matrix that is based on the computation of a saltation matrix that takes into account how the impact times evolve with the initial conditions. This closed-form formula is based on the assumption that the impacts are *simple impacts* in the sense that only one particle impacts at a given time. Moreover we consider non-grazing impacts, i.e. impact at nonzero velocities.

The case of a simple impact at time $t_\star > t_0$:

Let us assume that we have a unique and simple impact in the interval (t_0,t) at time $t_\star(Y_0)$. The notation outlines its dependency on the initial condition. At the impact time $t_\star(Y_0)$, the trajectory is reset using the elastic Newton impact law that can be written as follows

$$Y(t_\star^+(Y_0)) = R_{t_\star} Y(t_\star^-(Y_0)) \tag{24}$$

where $R_{t_\star} \in \mathbb{R}^{2p\times 2p}$ is the reset matrix. Let us denote by $i_{t_\star}$ the index of the impacting particle at $t_\star(Y_0)$, i.e.

$$y_{i_{t_\star}}(t_\star(Y_0)) = -1. \tag{25}$$

The reset matrix can be written as

$$R_{t_\star} = \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \tag{26}$$

where the matrix $E \in \mathbb{R}^{p\times p}$ is given by its components as

$$E_{ij} = \begin{cases} 0, \text{ if } i \neq j, \\ 1, \text{ if } i = j \neq i_{t_\star}, \\ -1, \text{ if } i = j = i_{t_\star}. \end{cases} \tag{27}$$

The state of the system at time t can be written as

$$\begin{aligned} Y(t) &= \phi(t,t_0,Y_0) = \phi(t,t_\star^+(Y_0),Y(t_\star^+(Y_0))) \\ &= \phi(t,t_\star^+(Y_0),R_{t_\star}Y(t_\star^-(Y_0))) = \phi(t,t_\star^+(Y_0),R_{t_\star}\phi(t_\star^-(Y_0),t_0,Y_0)). \end{aligned} \tag{28}$$

The differentiation of the previous expression amounts to differentiating with respect to Y_0 a composition of smooth functions

$$\begin{aligned} \frac{d\phi(t,t_0,Y_0)}{dY_0} &= D_2\phi(t,t_\star^+(Y_0),R_{t_\star}\phi(t_\star^-(Y_0),t_0,Y_0))\frac{dt_\star(Y_0)}{dY_0} \\ &\quad + D_3\phi(t,t_\star^+(Y_0),R_{t_\star}\phi(t_\star^-(Y_0),t_0,Y_0))R_{t_\star}\frac{d\phi(t_\star^-(Y_0),t_0,Y_0)}{dY_0} \end{aligned} \tag{29}$$

with

$$\frac{d\phi(t_\star^-(Y_0),t_0,Y_0)}{dY_0} = D_1\phi(t_\star^-(Y_0),t_0,Y_0)\frac{dt_\star(Y_0)}{dY_0} + D_3\phi(t_\star^-(Y_0),t_0,Y_0). \tag{30}$$

The notation $D_k\phi$ denotes the partial derivatives of ϕ with respect to its k-th argument. If the smooth flow is known between impacts, the only difficult part that remains to compute is the derivative of the time of impact $t_\star$ with respect to Y_0. Let us split the flow ϕ such that

$$Y(t) = \phi(t,t_0,Y_0) = \begin{bmatrix} \phi_y(t,t_0,Y_0) \\ \phi_{\dot{y}}(t,t_0,Y_0) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}. \tag{31}$$

We have assumed that only one particle of index $i_{t_\star}$ is impacting at $t_\star(Y_0)$. The constraint (25) can be written as

$$\phi_{y,i_{t_\star}}(t_\star,t_0,Y_0) = -1. \tag{32}$$

Since $\partial_t\phi_{y,i_{t_\star}}(t_\star^-,t_0,Y_0) = \dot{y}_{i_{t_\star}}(t_\star^-(Y_0)) < 0$ (non-grazing impact) and the flow is smooth (analytic) between impacts, the implicit function theorem guarantees that the impact persists upon small variations of Y_0, with an impact time $t_\star$ being a smooth (analytic) function of Y_0. Moreover, defining a projection matrix $P_i \in \mathbb{R}^{1\times 2p}$ such that

$$D_3\phi_{y,i}(t_\star^-(Y_0),t_0,Y_0) = P_i D_3\phi(t_\star^-(Y_0),t_0,Y_0), \tag{33}$$

we have

$$\frac{dt_{t_\star}(Y_0)}{dY_0} = -\frac{1}{\dot{y}_{i_{t_\star}}(t_\star^-(Y_0))} P_{i_{t_\star}} D_3\phi(t_\star^-(Y_0),t_0,Y_0). \tag{34}$$

In order to simplify the expression of the monodromy matrix given by (29) and (30), we observe that

$$D_2\phi(t,t_\star^+,Y(t_\star^+(Y_0))) = -D_3\phi(t,t_\star^+(Y_0),Y(t_\star^+(Y_0)))\dot{Y}(t_\star^+(Y_0)). \tag{35}$$

Indeed, since $\phi(t,\tilde{t},\phi(\tilde{t},t_\star^+,Y_\star)) = \phi(t,t_\star^+,Y_\star)$ is independent of $\tilde{t}$, the identity $\partial_{\tilde{t}}\phi(t,\tilde{t},\phi(\tilde{t},t_\star^+,Y_\star)) = 0$ evaluated at $\tilde{t} = t_\star^+$ and $Y_\star = Y(t_\star^+(Y_0))$ yields identity (35). Using (29), (30) and (35), the monodromy matrix simplifies to

$$\frac{d\phi(t,t_0,Y_0)}{dY_0} = D_3\phi(t,t_\star^+,Y(t_\star^+(Y_0))) \left[[R_{t_\star}\dot{Y}(t_\star^-(Y_0)) - \dot{Y}(t_\star^+(Y_0))]\frac{dt_\star(Y_0)}{dY_0} + R_{t_\star}D_3\phi(t_\star^-(Y_0),t_0,Y_0) \right]. \tag{36}$$

Finally, using the relation (34), the monodromy matrix is expressed as follows

$$\frac{d\phi(t,t_0,Y_0)}{dY_0} = D_3\phi(t,t_\star^+(Y_0),Y(t_\star^+(Y_0)))S_{t_\star}D_3\phi(t_\star^-(Y_0),t_0,Y_0), \quad t > t_\star(Y_0) \tag{37}$$

where the so-called saltation matrix $S_{t_\star}$ is defined by

$$S_{t_\star} = -\frac{1}{\dot{y}_{i_{t_\star}}(t_\star^-(Y_0))}\left[R_{t_\star}\dot{Y}(t_\star^-(Y_0)) - \dot{Y}(t_\star^+(Y_0))\right]P_{i_{t_\star}} + R_{t_\star}. \tag{38}$$

Note that the monodromy matrix is obtained as the product of the Jacobian matrices of the flow with respect to the initial condition in each smooth phases separated by the saltation matrix.

The case of two simple impacts at times $t_{\star,2} > t_{\star,1} > t_0$:

For the two simple impacts at time $t_{\star,2} > t_{\star,1} > t_0$, the computation of the monodromy matrix follows the same line. It is also a product of the Jacobian matrices of the flow with respect to the initial condition in each smooth phases separated by the saltation matrix:

$$\begin{aligned}\frac{d\phi(t,t_0,Y_0)}{dY_0} &= D_3\phi(t,t_{\star,2}^+(Y_0),Y(t_{\star,2}^+(Y_0)))S_{t_{\star,2}}\\ &\quad D_3\phi(t,t_{\star,1}^+(Y_0),Y(t_{\star,1}^+(Y_0)))S_{t_{\star,1}}D_3\phi(t_{\star,1}^-(Y_0),t_0,Y_0), \quad t > t_{\star,2}(Y_0).\end{aligned} \tag{39}$$

Computation of the monodromy for the piecewise linear system :

In our case of a piecewise-linear dynamics, the flow of the system between two impacts is given by

$$\phi(t,t_0,Y_0) = \exp(D(t-t_0))Y_0, t_0 \le t \le t_{\star,1}(Y_0), \tag{40}$$

$$\phi(t,t_{\star,1}^+(Y_0),Y(t_{\star,1}^+(Y_0))) = \exp(D(t-t_{\star,1}(Y_0)))Y(t_{\star,1}^+(Y_0)), t_{\star,1}(Y_0) \le t \le t_{\star,2}(Y_0) \tag{41}$$

$$\phi(t,t_{\star,2}^+(Y_0),Y(t_{\star,2}^+(Y_0))) = \exp(D(t-t_{\star,2}(Y_0)))Y(t_{\star,2}^+(Y_0)), t \ge t_{\star,2}(Y_0), \tag{42}$$

with $D = J + \gamma L$. As indicated above in the derivation of the monodromy matrix, the piecewise linear flow is smooth (analytic). If we consider consider the explicit formula of the linear flow (40),(41),(42) between impacting times at $t_{\star,1} = T/2$ and $t_{\star,2} = T$, we get for the monodromy matrix

$$\frac{d\phi(t,t_0,Y_0)}{dY_0} = \exp(D(t-T))\,S_T\,\exp(D(T/2))\,S_{T/2}\,\exp(D(T/2-t_0)), \quad t > T, \tag{43}$$

where $t_0 < T/2$. In section 3, we shall fix $t_0 = T/4$ and $t = t_0 + T = 5T/4$ to compute the monodromy matrix of a T-periodic solution with impact times multiple of $T/2$. This leads to

$$\frac{d\phi(5T/4,T/4,Y_0)}{dY_0} = \exp(DT/4)\,S_T\,\exp(D(T/2))\,S_{T/2}\,\exp(DT/4). \tag{44}$$

The periodic solution will be unstable if this monodromy matrix admits an eigenvalue with modulus greater than unity, and spectrally stable if all eigenvalues lie on the unit circle (due to time-reversal symmetry, the Floquet spectrum has the invariance $\sigma \to \sigma^{-1}$). The spectrum of the above monodromy is the same as for $S_T \exp(D(T/2))\, S_{T/2} \exp(DT/2)$.

3 Numerical computation of nonsmooth modes

We solve problem (12)-(13) numerically for a chain of p oscillators with periodic boundary conditions. Unless explicitly stated otherwise, we fix $p = 100$. Although the system (12)-(13) is a standard linear system, we use a general shooting method, i.e. determine a vector $\xi = (y^{(0)}(0), y^{(1)}(0), \dot{y}^{(2)}(0)) \in \mathbb{R}^p$ such that the three boundary conditions of (13) at $t = 0$ and $t = T/2$ are satisfied through Newton iterations. For each Newton iteration, this requires to solve a linear system for ξ obtained through time-integration of the linear ODE (12). This time integration is equivalent to computing numerically the exponential matrix of the linear flow. When the coupling parameter is chosen far from the degeneracy case of the BVP matrix, the shooting technique converges in one iteration. When we are in the neighborhood of the degenerate cases, the number of Newton iterations may increase indicating an ill-conditioned linear system of the BVP. Thanks to the general shooting technique, the case of nonlinear local or interaction potentials could be addressed similarly. The constraint (14) is checked *a posteriori*. To this end, we integrate (1)-(2)-(3) numerically using an event-driven scheme for nonsmooth dynamical systems implemented in the Siconos software [43]. For the shooting technique and validation of the constraints, the linear ODE is integrated thanks to ODEPACK [21] embedded in the Siconos software.

Usually, the solution branches are first continued for fixed values of T, varying the coupling parameter γ. For all fixed value $T \in (\pi, 2\pi)$, a choice of impacting particles and phases (determined by I_1, I_2) selects a unique solution for $\gamma = 0$ which can be continued up to some maximal value of the coupling parameter γ. We shall see in the sequel that some continuations are also done with respect to the period.

3.1 Site-centered breathers

In this section, we illustrate the site-centered breather for the mode pattern $I_2 = \{50\}, I_1 = \emptyset$ depicted in Figure 6. The period is $T = \frac{3\pi}{2}$. The periodic solution has been successfully computed for $\gamma \in [0, \gamma_c]$ with

$$\gamma_c = \frac{1}{4}((\frac{2\pi}{T})^2 - 1), \tag{45}$$

the critical value of γ for which we expect to reach the out-of-phase mode. For $T = \frac{3\pi}{2}$, we have $\gamma_c \approx 0.1944$. In Figure 7, the initial positions and velocities are displayed for the particle indices between 40 and 60 and for 4 different values of γ. We observe that for small values of the coupling parameter γ the breather is localized on a few particles. With the increasing values of γ, the support of the solution is increasing to reach the out-of-phase linear grazing mode for $\gamma = \gamma_c$. Let us note that the velocity of the central particle 50 is decreasing to the grazing solution for all the particles.

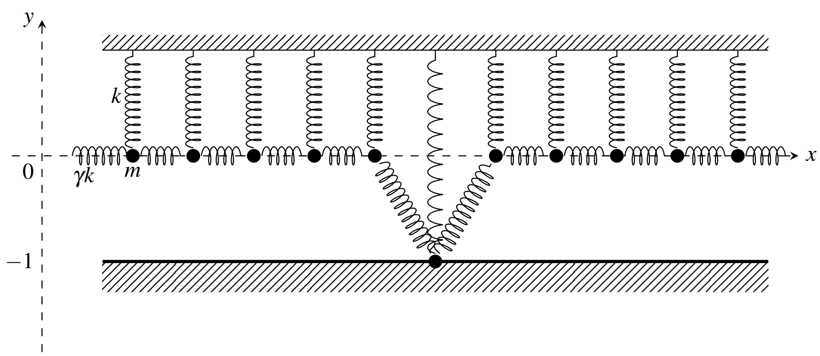


Fig. 6: Mode pattern for the site-centered breather

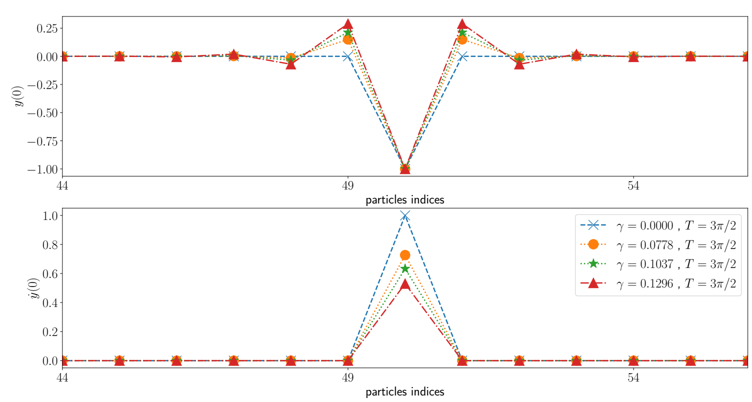
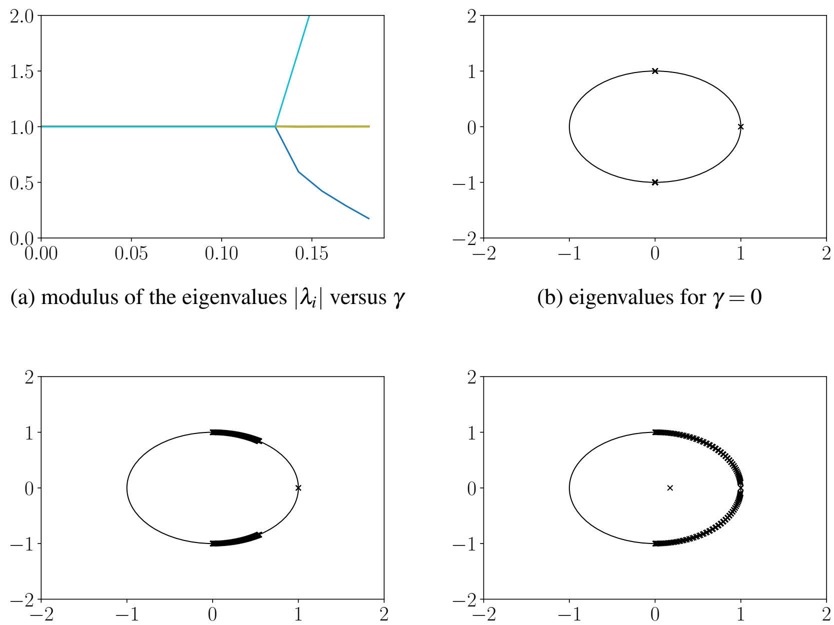


Fig. 7: Site-centered breather with pattern $I_1 = \emptyset, I_2 = \{50\}$

In Figure 8, the eigenvalues of the monodromy matrix are displayed. In Figure 8(a), we remark that the eigenvalues have a modulus equal to 1 up to a critical value γ_s between 0.129 and 0.142 for which a pair of eigenvalues is leaving the unit circle. In Figures 8(b), 8(c) and 8(d), all the eigenvalues are plotted in the complex plane for three different values of $\gamma \in \{0, 0.064, 0.181\}$. For $\gamma = 0$, a pair of eigenvalues are equal to $+1$ and all the other conjugate eigenvalues pairs are equal to i or $-i$. For $\gamma < \gamma_s$, the conjugate eigenvalues pairs, equals to i and $-i$ for $\gamma = 0$, start to slide on the unit circle toward the pair of eigenvalues that remains at $+1$. For $\gamma = \gamma_s$, a collision occurs in $+1$. Finally, for $\gamma > \gamma_s$, a pair of real inverse eigenvalues is leaving the unit circle to slide on the real line while a pair of eigenvalues is remaining at $+1$. In that case, the stability of the periodic solution is lost. For $\gamma = 0.181$, one the eigenvalues of modulus around 5.71 is not displayed. To illustrate this loss of stability, we report in Figure 9 several time integrations of the system with constraints and impacts for different values of γ over the time interval $[0, 25T]$. Although the system is numerically integrated with high accuracy Runge-Kutta schemes in ODEPACK with very tight tolerances (10^{-14}), the periodic solutions for $\gamma = 0.181$ is destabilized.



(a) modulus of the eigenvalues $|\lambda_i|$ versus γ

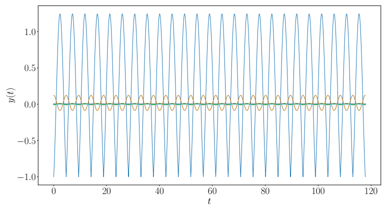
(b) eigenvalues for $\gamma = 0$

(c) eigenvalues for $\gamma = 0.064$

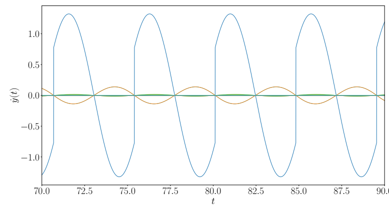
(d) eigenvalues for $\gamma = 0.181$

Fig. 8: Eigenvalues of the monodromy matrix for the site-centered breather with pattern $I_1 = \emptyset, I_2 = \{50\}$

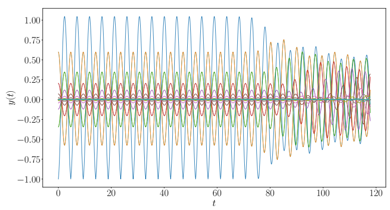
We also perform a continuation of the solution with respect to the period. We start for a value of (γ, T) equals to $(0.15, 3\pi/2)$ and we decrease the period following a



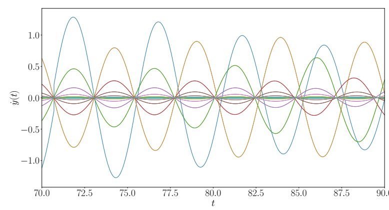
(a) positions of the particles for $\gamma = 0.064$



(b) velocities of the particles for $\gamma = 0.064$



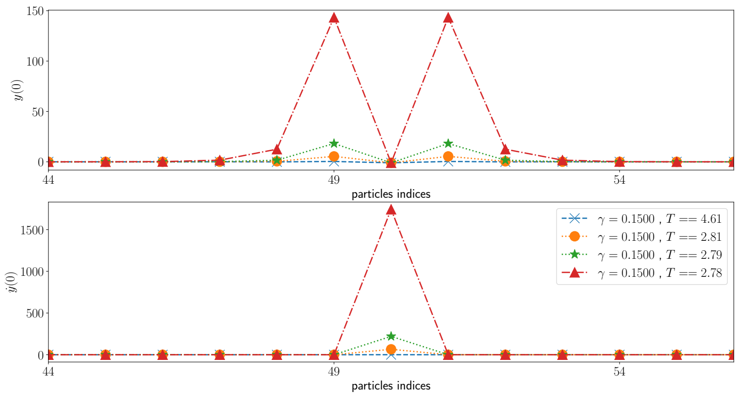
(c) positions of the particles for $\gamma = 0.181$



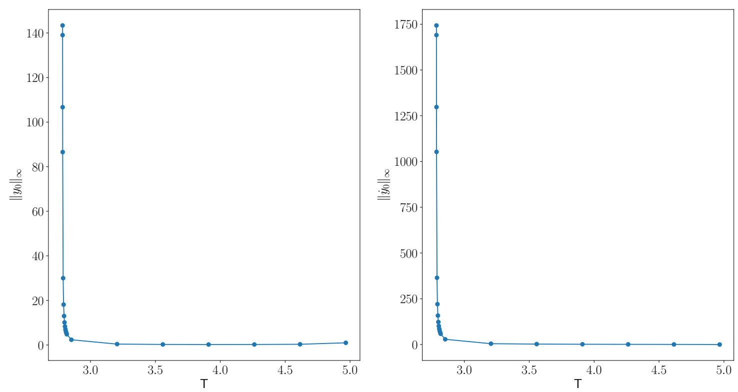
(d) velocities of the particles for $\gamma = 0.181$

Fig. 9: Time integration of the periodic solutions for the site-centered breather with pattern $I_1 = \emptyset, I_2 = \{50\}$

solution with a fixed pattern. The numerical solutions are displayed in Figure 10(a). We can observe that a family of site-centered breathers is found with an increasing amplitude of the initial state. For the uncoupled case ($\gamma = 0.0$), we know that the amplitude of the solution goes to infinity when $T \to \pi$. The same phenomenon is observed for a given coupling parameter $\gamma = 0.15$. In Figure 10(b), we plot the maximum amplitude of the position $\|y(0)\|_\infty$ and the velocity $\|\dot{y}(0)\|_\infty$ as a function of T. An asymptotic value of the period appears clearly for which the amplitude of the solution blows up. In this specific case, the asymptotic value of the period is about $0.58(3\pi/2) \approx 2.78$. Let us note that this value is below π.



(a) positions and velocities of the particles



(b) maximum amplitude of the position $\|y(0)\|_\infty$ and the velocity $\|\dot{y}(0)\|_\infty$ as a function of T

Fig. 10: Continuation with a decreasing period of the site-centered breather with pattern $I_1 = \emptyset, I_2 = \{50\}$ for $\gamma = 0.15$.

To conclude this section, an exploration of the viability of the site-centered breathers has been performed for $(\gamma, T) \in [0, 1.1] \times [2, 2\pi]$ and $p = 30$ particles. We select a mesh grid in the plane (γ, T) and solve the boundary value problem for each pair (γ, T). The results are reported in Figure 11. The light areas correspond to a numerical computation of a periodic solution of (12)-(13) with the satisfaction of the constraint (14) and the pattern $I_1 = \emptyset, I_2 = \{15\}$. The red dashed curve is given by the out-of-phase grazing linear mode whose period is related to γ by

$$T(\gamma) = 2\pi\,(1+4\gamma)^{-1/2}. \tag{46}$$

As expected with the previous computations, we observe that there exists a large light area bounded above by the relation (46) and corresponding to site-centered breathers. This area is also bounded below by a another curve that corresponds to modes whose amplitudes go to infinity as we have already discussed for a particular value of $\gamma = 0.15$ in Figure 10. Quite interestingly, other light areas are present above the red curve. To explain these areas, we plot the graphs of the periods with respect to γ for larger wavenumber q given by

$$T_n(\gamma) = 2\pi\,\left(1+4\gamma\sin^2(q/2)\right)^{-1/2}, \text{ with } q = n2\pi/p, \quad n = 1,\dots,15. \tag{47}$$

We can observe the existence of modulated waves near the linear grazing solutions. In order to illustrate the solutions obtained in these areas, we plot in Figure 12 the results of two continuations over the period for $\gamma = 1$, $T_3 \approx 5.34$ and $T_4 \approx 4.87$ (large dots in Figure 11). We can observe that these solutions are not exactly normal nonsmooth modes that emerge from the linear grazing modes but rather spatial modulations of nonsmooth normal modes. For the computation of what could be called a *nonsmooth normal mode* we refer to Section 3.4. There, other solutions are computed (with long-wavelength near T_1) with preservation of the normal mode pattern at the start of continuation.

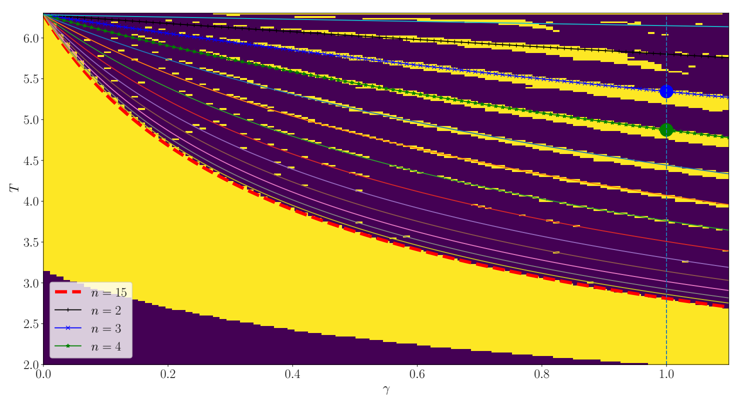
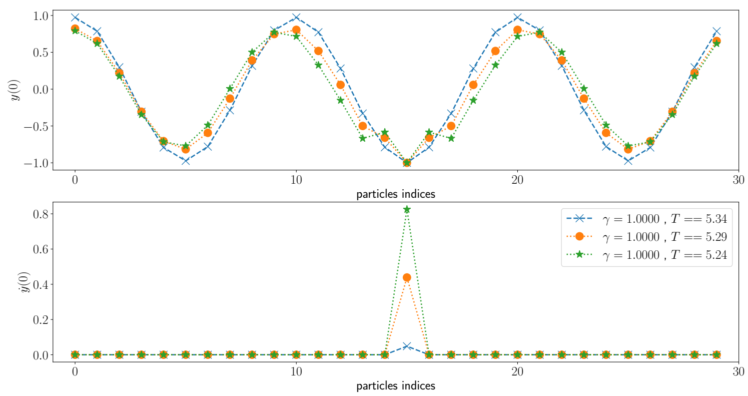
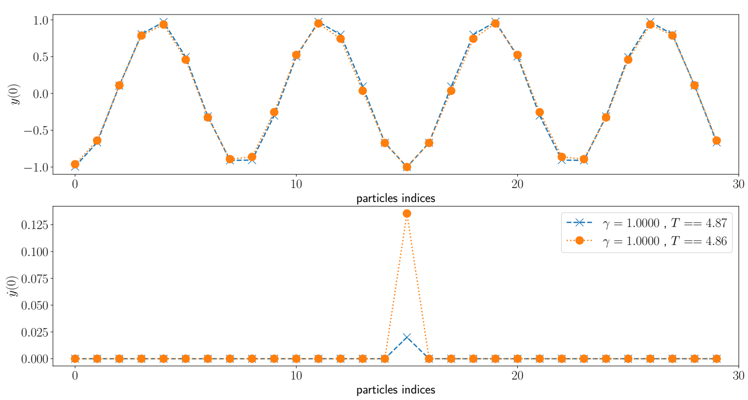


Fig. 11: Continuation of periodic solutions with pattern $I_1 = \emptyset, I_2 = \{15\}$ (light areas) for $(\gamma, T) \in [0, 1.1] \times [2, 2\pi]$. Graphs of $T_n(\gamma) = 2\pi\,\left(1+4\gamma\sin^2(q/2)\right)^{-1/2}$, with $q = n2\pi/p$, for $n = 1,\dots,15$ and $p = 30$.



(a) Continuation for the value of the period around $T_3 = 2\pi\left(1+4\gamma\sin^2(3\pi/30)\right)^{-1/2} \approx 5.34$



(b) Continuation for the value of the period around $T_4 = 2\pi\left(1+4\gamma\sin^2(4\pi/30)\right)^{-1/2} \approx 4.87$

Fig. 12: Continuation of spatially-modulated nonsmooth normal modes with pattern $I_1 = \emptyset, I_2 = \{15\}$ for $\gamma = 1$.

3.2 Bond-centered breathers

In this section, some bond-centered breathers are computed with two different patterns.

Bond-centered breathers with pattern $I_1 = \{49\}, I_2 = \{50\}$

Let us start with the out-of-phase pattern $I_1 = \{49\}, I_2 = \{50\}$ illustrated in Figure 13. We choose again a period equal to $\frac{3\pi}{2}$ and the periodic solution has successfully been computed in the range $[0, \gamma_c]$ with γ_c given by (45). The initial conditions of the periodic solutions are displayed in Figure 14 for the particle indices in $[40, 60]$. Again, we can observe that the breather is localized over few particles for small values of the coupling parameter. Once again, the solution reaches the out-of-phase linear grazing mode for $\gamma = \gamma_c$ while the velocity of the central particle decreases at time 0.

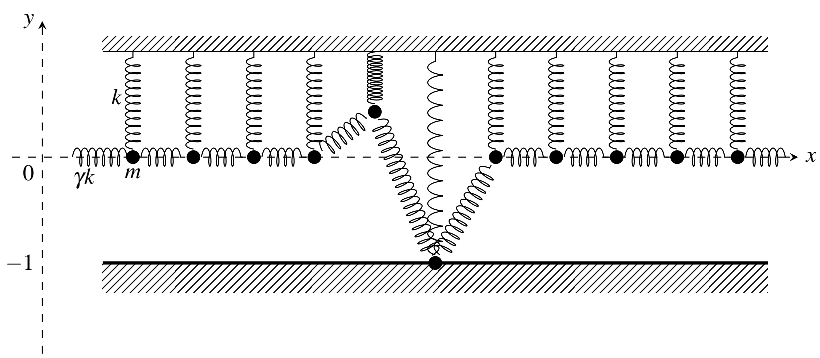


Fig. 13: Mode pattern for the bond-centered breather

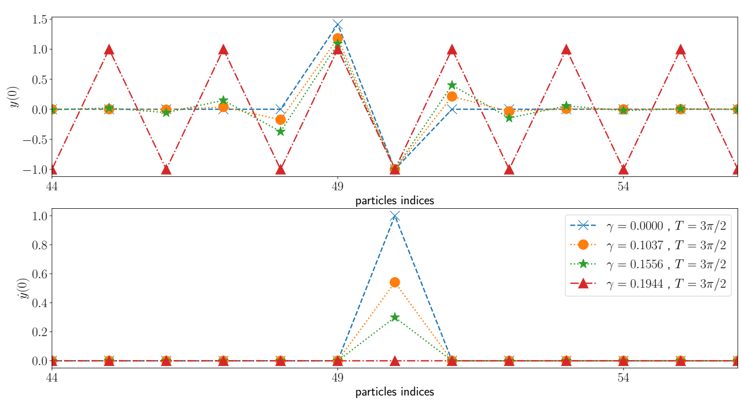
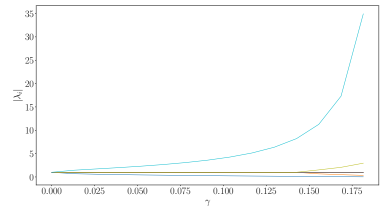
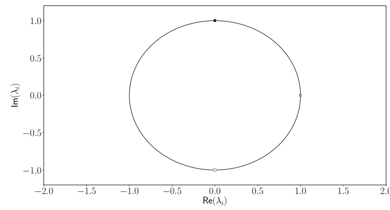


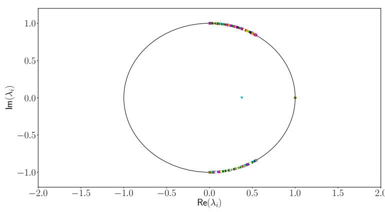
Fig. 14: Bond-centered breather with pattern $I_1 = \{49\}, I_2 = \{50\}$



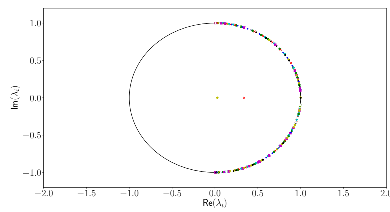
(a) modulus of the eigenvalues $|\lambda_i|$ versus γ



(b) eigenvalues for $\gamma = 0$

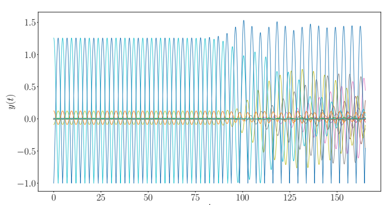


(c) eigenvalues for $\gamma = 0.064$

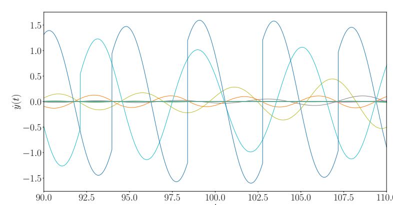


(d) eigenvalues for $\gamma = 0.181$

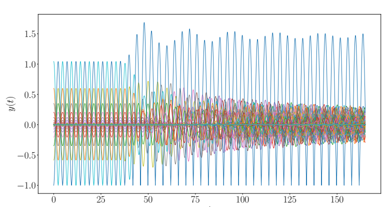
Fig. 15: Eigenvalues of the monodromy matrix for the bond-centered breather with pattern $I_1 = \{49\}, I_2 = \{50\}$



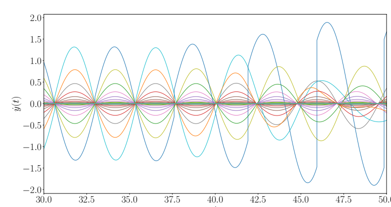
(a) positions of the particles for $\gamma = 0.064$



(b) velocities of the particles for $\gamma = 0.064$



(c) positions of the particles for $\gamma = 0.181$



(d) velocities of the particles for $\gamma = 0.181$

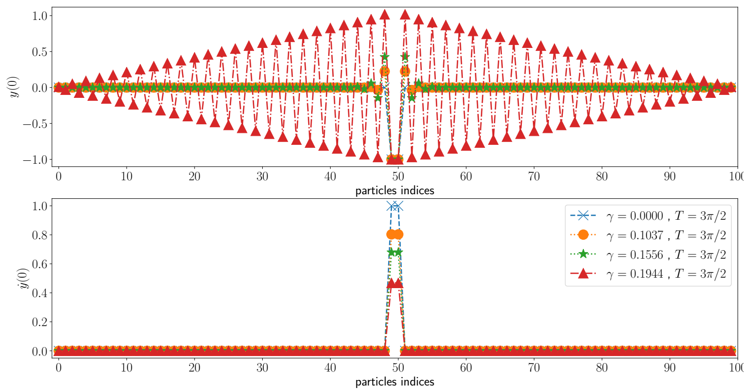
Fig. 16: Time integration of the periodic solutions for the bond-centered breather with pattern $I_1 = \{49\}, I_2 = \{50\}$

In Figure 15, we depict the eigenvalues of the monodromy matrix. In Figure 15(b) for $\gamma = 0$, we have two pairs of eigenvalues in $+1$. All the other pairs of conjugate eigenvalues are equal to i or $-i$. We observe in Figure 15(a) and Figure 15(b) that for $\gamma > 0$, a pair of real inverse eigenvalues slides from $+1$ on the real line as γ increases while the other pair remains equal to $+1$. The others pairs of conjugate eigenvalues slide on the unit circle toward the pair of real eigenvalues in $+1$. A collision occurs again at $+1$ for $\gamma = \gamma_s \in [0.142, 0.155]$. Then, a second pair of inverse real eigenvalues slides on the real line. For $\gamma > 0$, the stability of the periodic solutions is lost. We attempt to illustrate this phenomena with numerical time integration of the periodic solutions over a long time interval $[0, 35T]$ in Figure 16.

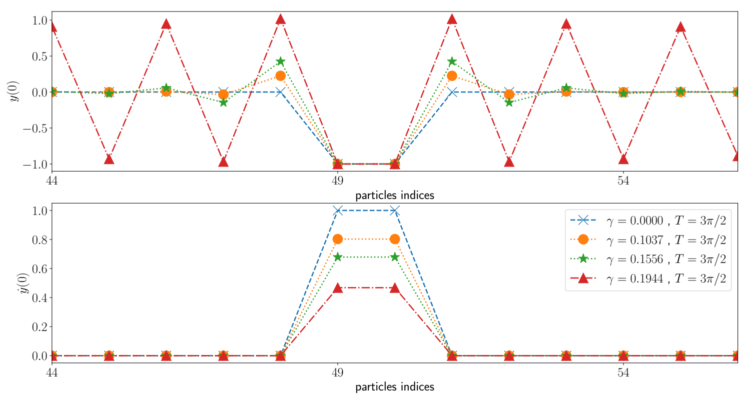
Bond-centered breathers with pattern $I_1 = \emptyset, I_2 = \{49, 50\}$

For the pattern $I_1 = \emptyset, I_2 = \{49, 50\}$, the solution for the initial conditions is depicted for the whole chain in Figure 17(a) and for the particles with indices in $[40, 60]$ in Figure 17(b). The period is again $\frac{3\pi}{2}$ and we succesfully perform a continuation of the solution over $[0, \gamma_c]$ with γ_c given by (45). The main difference with the previous breathers concerns the solution when $\gamma \to \gamma_c$. In this latter case, it seems that we do not converge towards a grazing linear mode. This has to be confirmed with a more accurate study of the critical value of γ.

In Figure 18, we depict the eigenvalues of the monodromy matrix computed by finite differences. In this case, the closed form formula of the monodromy (44) does no longer apply since we have multiple impacts. Although the approximation of the eigenvalues may contain some numerical errors, we observe a more complicated behavior of the evolution with respect to γ of the eigenvalues. For $\gamma = 0$, two pairs of real eigenvalues are equal to $+1$ and the others are conjugated pairs of eigenvalues are equal to i and $-i$. For increasing values of γ, one of the pairs of real eigenvalues starts to slide on the unit circle respectively towards i and $-i$ while the other pairs of conjugate eigenvalues slide on the unit circle from i and $-i$ towards $+1$. A first collision occurs on the unit circle for $\gamma \in [0.051, 0.064]$ and two pairs of eigenvalues leave the unit circle. Several other collisions of different types occur when we increase the value of γ up to γ_c.

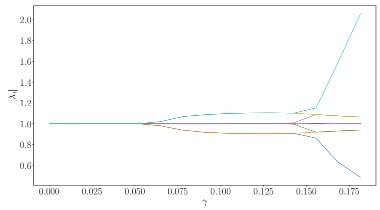


(a) initial positions and velocities of the particles

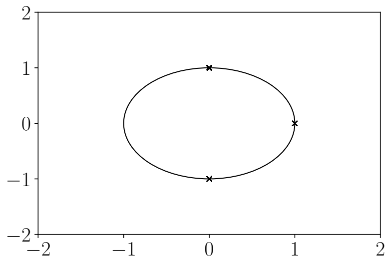


(b) initial positions and velocities of the particles with indices in $[40,60]$

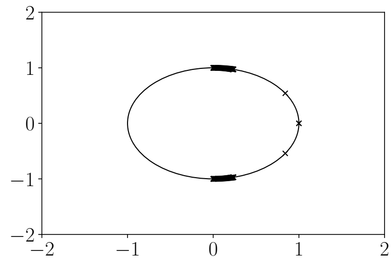
Fig. 17: Bond-centered breather with pattern $I_1 = \emptyset, I_2 = \{49,50\}$



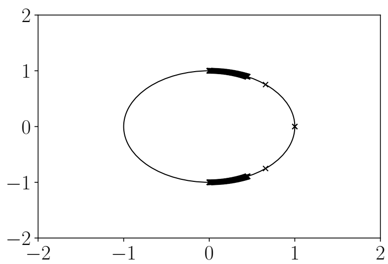
(a) modulus of the eigenvalues $|\lambda_i|$ versus γ



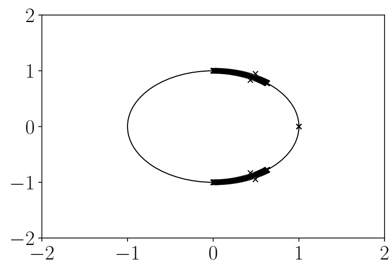
(b) eigenvalues for $\gamma = 0$



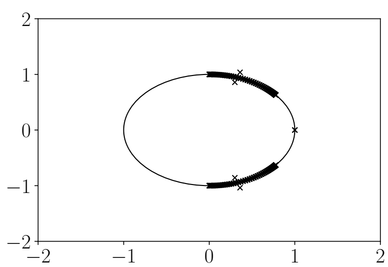
(c) eigenvalues for $\gamma = 0.025$



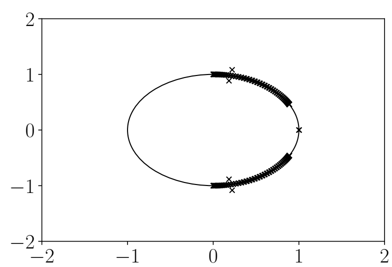
(d) eigenvalues for $\gamma = 0.051$



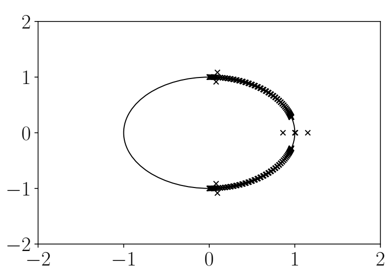
(e) eigenvalues for $\gamma = 0.077$



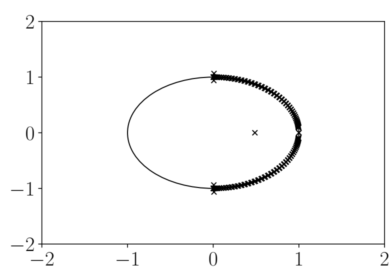
(f) eigenvalues for $\gamma = 0.103$



(g) eigenvalues for $\gamma = 0.129$



(h) eigenvalues for $\gamma = 0.155$



(i) eigenvalues for $\gamma = 0.181$

Fig. 18: Eigenvalues of the monodromy matrix computed by finite differences for the bond-centered breather with pattern $I_1 = \emptyset, I_2 = \{49, 50\}$

3.3 Multiple impacting particles

In this section, we illustrate wave patterns with multiple impacts, where the pattern is either spatially periodic or localized on several particles (multi-site breathers).

Out-of-phase mode with spatial period two

We start with the nonsmooth mode of spatial period two described in Section 2. The pattern is given by $I_1 = \{2k+1\}_{k=0,\ldots,49}, I_2 = \{2k\}_{k=0,\ldots,49}$ that corresponds to the sets of odd and even integers respectively. In Figure 19, the initial conditions for the periodic solutions are given for $T = \frac{3\pi}{2}$. For this example, we are able to continue the solution over the range $[0, \gamma_c]$ up to reaching the out-of-phase linear grazing mode. In Figure 20, the eigenvalues of the monodromy matrix computed by finite differences are depicted. For $\gamma = 0$, all the eigenvalues are equal to $+1$. For $\gamma > 0$, the pairs of inverse real eigenvalues slide on the real line. The periodic solutions are therefore unstable for $\gamma > 0$. This is illustrated in Figure 21 where long time integration simulations have been performed over the time interval $[0, 35T]$.

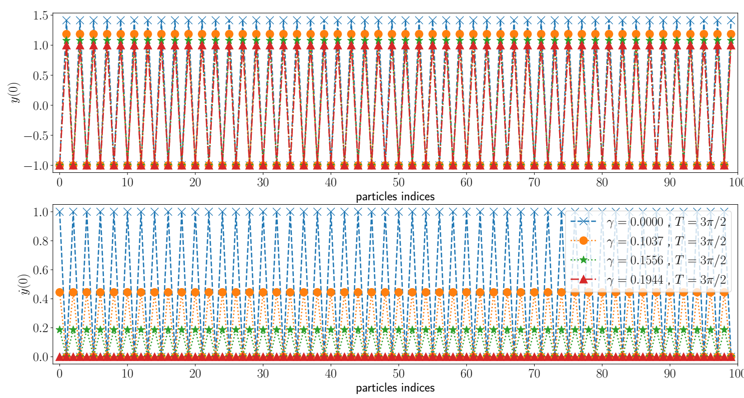
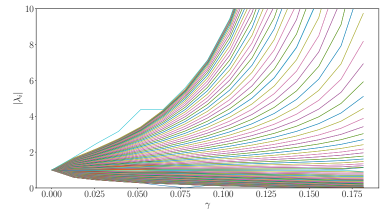


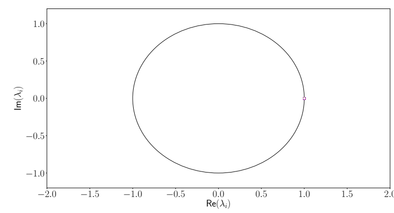
Fig. 19: Out-of-phase mode with pattern $I_1 = \{2k+1\}_{k=0,\ldots,49}, I_2 = \{2k\}_{k=0,\ldots,49}$

Periodic wave with spatial period six

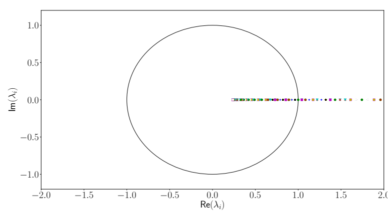
Another example of nonsmooth spatially periodic standing wave is displayed in Figure 22. The spatial period is six and the time period is again $\frac{3\pi}{2}$. The mode profiles are depicted for several values of γ in $[0, \gamma_c]$.



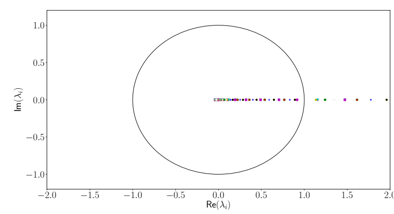
(a) modulus of the eigenvalues $|\lambda_i|$ versus γ



(b) eigenvalues for $\gamma = 0$

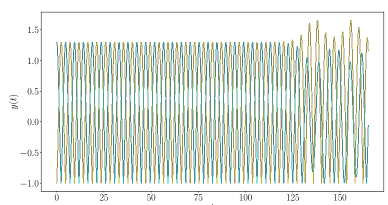


(c) eigenvalues for $\gamma = 0.064$

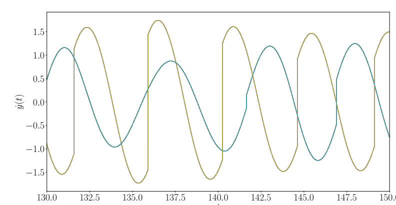


(d) eigenvalues for $\gamma = 0.181$

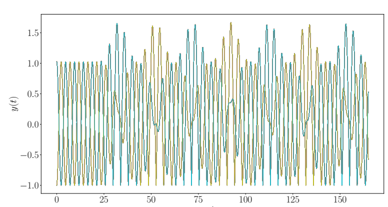
Fig. 20: Eigenvalues of the monodromy matrix computed by finite differences for the out-of-phase mode with pattern $I_1 = \{2k+1\}_{k=0,\ldots,49}, I_2 = \{2k\}_{k=0,\ldots,49}$



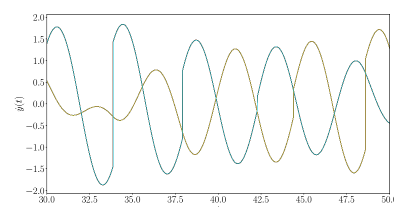
(a) positions of the particles for $\gamma = 0.051$



(b) velocities of the particles for $\gamma = 0.051$



(c) positions of the particles for $\gamma = 0.181$



(d) velocities of the particles for $\gamma = 0.181$

Fig. 21: Time integration of the periodic solutions for the out-of-phase mode with pattern $I_1 = \{2k+1\}_{k=0,\ldots,49}, I_2 = \{2k\}_{k=0,\ldots,49}$

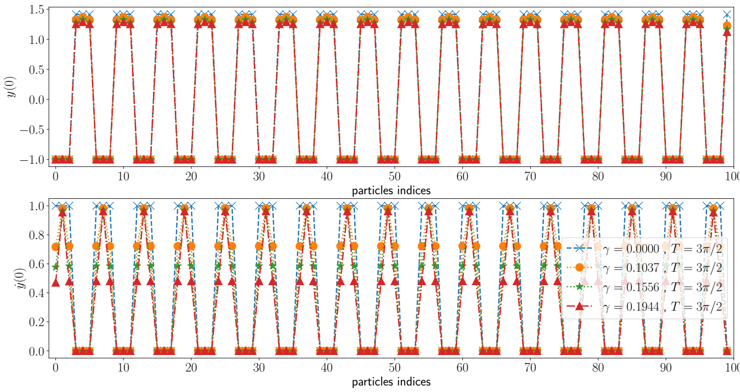


Fig. 22: Periodic wave with pattern of spatial period 6 : $I_1 = \{6k+3, 6k+4, 6k+5\}_{k=0,3,\ldots}, I_2 = \{6k, 6k+1, 6k+2\}_{k=0,3,\ldots}$

Multi-site breather localized on 10 particles

In Figure 23, a multi-site breather with pattern $I_1 = \emptyset, I_2 = \{45,\ldots,55\}$ is displayed for $T = \frac{3\pi}{2}$. For $\gamma \to \gamma_c$, the computation of the solutions is more difficult. The largest value of γ for which a solution is displayed is $0.1944096 < \gamma_c$. We can observe that the particles in I_0 are still not grazing.

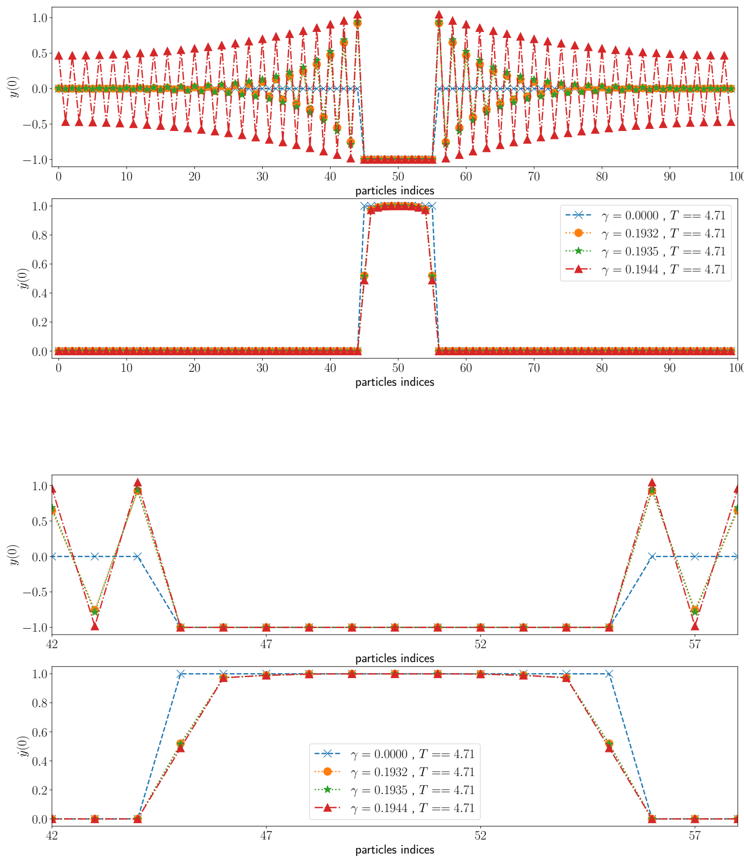


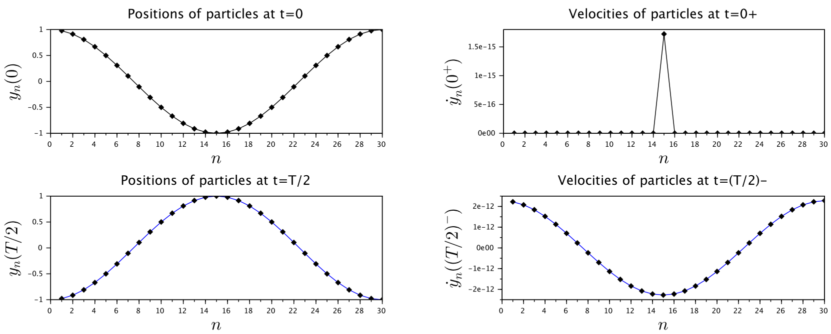
Fig. 23: Multi-site breather with pattern $I_1 = \emptyset, I_2 = \{45, \dots, 55\}$

3.4 Long-wavelength modes

We also compute spatially extended long-wavelength modes close to the main linear mode with wavenumber $q = 2\pi/p$ that is depicted in Figure 24. The period of the linear mode for a given wavenumber q is

$$T_1 = 2\pi \left(1 + 4\gamma \sin^2(q/2)\right)^{-1/2}. \tag{48}$$

Our computations are performed for $\gamma = 1$ and $p = 30$ particles and we get $T_1 \approx 6.150$.

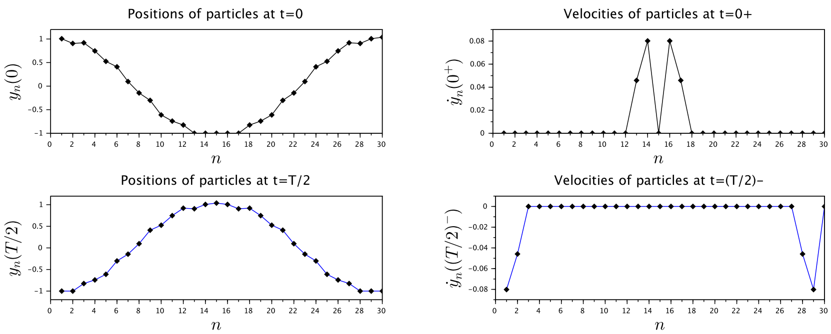


(a) Positions for $T = T_1$ (b) Velocities for $T = T_1$

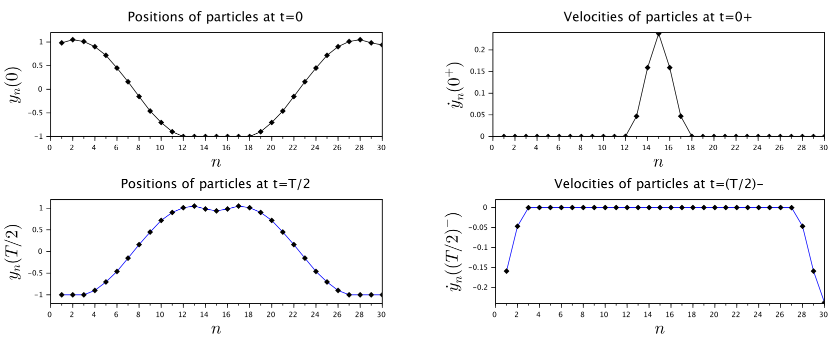
Fig. 24: Main linear grazing mode for $\gamma = 1$ and $T_1 = 2\pi\,(1+4\gamma\sin^2(\pi/30))^{-1/2}$.

A first branch of solutions

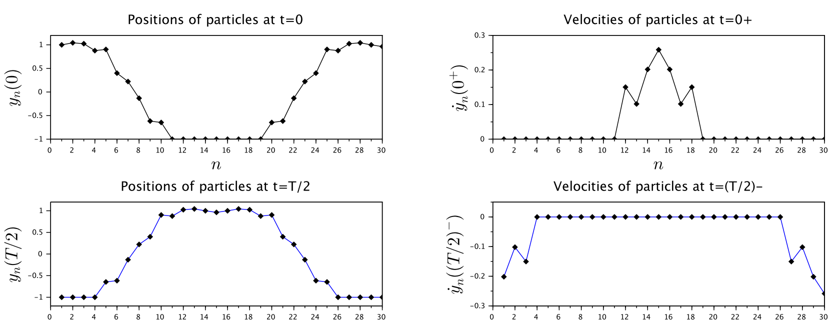
We are able to follow a first continuous branch of solutions depicted in Figure 25 with periods $T \in [\alpha_7\,T_1, \alpha_1\,T_1]$, and $\alpha_1 = 0.99056$ and $\alpha_7 = 0.5035988$. The mode amplitude diverges when $T \to \alpha_7\,T_1^+$, and two particles at $n = 15, 30$ (the antinodes, i.e. the particles that reach maximal height) undergo grazing impacts when $T \to \alpha_1\,T_1^-$. The number of impacting particles decreases from 30 to 10 when T is increased. More precisely, for T in intervals of the form $[\alpha_j\,T_1, \alpha_{j-1}\,T_1]$, we find $4j+2$ impacting particles with pattern $I_1 = \{\,1,2,\ldots,j,p-j,p-j+1,\ldots,p\,\}$, $I_2 = \{\,15-j,\ldots,15+j\,\}$. We find $\alpha_6 \approx 0.5798$, $\alpha_5 \approx 0.7641$, $\alpha_4 \approx 0.92$, $\alpha_3 \approx 0.9618$, $\alpha_2 \approx 0.9771$.



(a) Positions for $T = \alpha_1 T_1$ (b) Velocities for $T = \alpha_1 T_1$

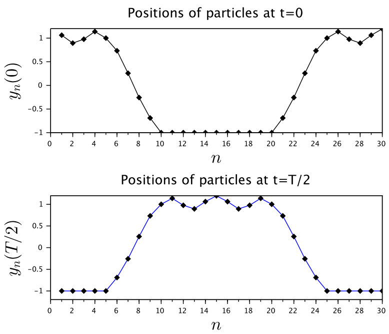


(c) Positions for $T = \alpha_2 T_1$ (d) Velocities for $T = \alpha_2 T_1$

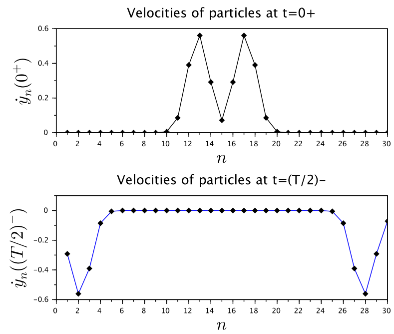


(e) Positions for $T = \alpha_3 T_1$ (f) Velocities for $T = \alpha_3 T_1$

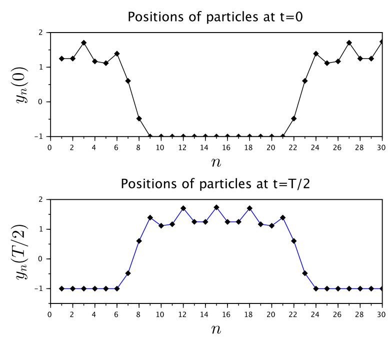
Fig. 25: A first branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi\,(1+4\gamma\sin^2(\pi/30))^{-1/2}$.



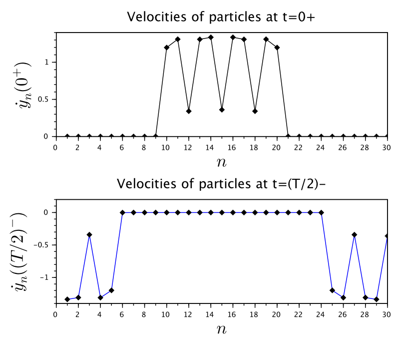
(g) Positions for $T = \alpha_4 T_1$



(h) Velocities for $T = \alpha_4 T_1$

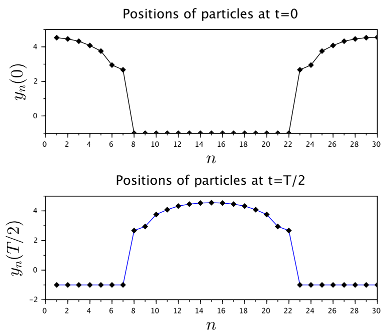


(i) Positions for $T = \alpha_5 T_1$

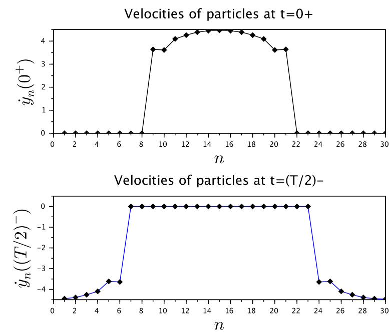


(j) Velocities for $T = \alpha_5 T_1$

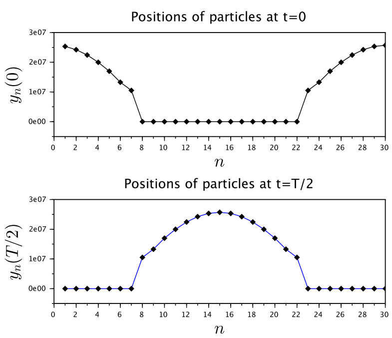
Fig. 25: A first branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi \left(1 + 4\gamma \sin^2(\pi/30)\right)^{-1/2}$ (continued).



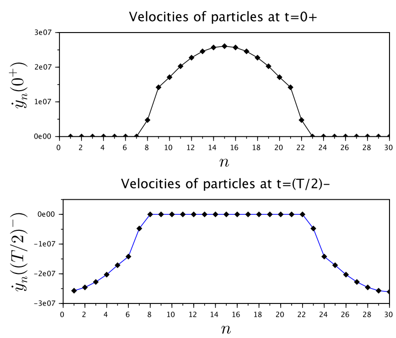
(k) Positions for $T = \alpha_6 T_1$



(l) Velocities for $T = \alpha_6 T_1$



(m) Positions for $T = \alpha_7 T_1$

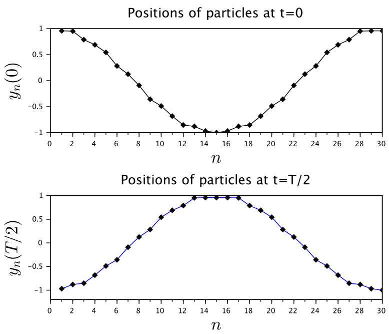


(n) Velocities for $T = \alpha_7 T_1$

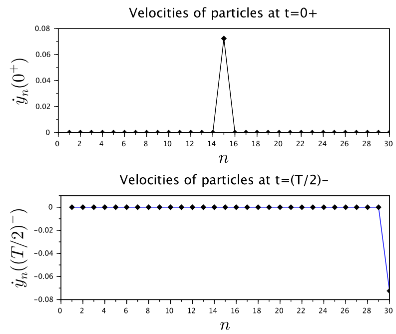
Fig. 25: A first branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi\,(1+4\gamma\sin^2(\pi/30))^{-1/2}$ (continued).

A second branch of solutions

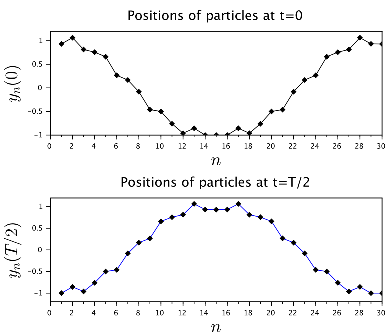
We find another branch of solutions whose period $T \in [0.81 \cdot T_1, T_1)$ can approach T_1 arbitrary closely. These solutions emerge from the linear grazing mode when $T \to T_1$. Let us set $T = \alpha T_1$ and describe the mode pattern depending on α. We only describe I_2, given that $I_1 = I_2 + 15 (\mathrm{mod}\, 30)$. We have $I_2 = \{15\}$ for $\alpha \in [0.991, 1)$, $I_2 = \{14, 15, 16\}$ for $\alpha \in [0.9825921, 0.99]$, $I_2 = \{12, 14, 15, 16, 18\}$ for $\alpha \in [0.965, 0.9825924]$, $I_2 = \{11, 12, 14, 15, 16, 18, 19\}$ for $\alpha \in [0.85, 0.964]$, $I_2 = \{9, 11, 12, 14, 15, 16, 18, 19, 21\}$ for $\alpha \in [0.836, 0.849]$, and for $\alpha \in [0.81, 0.835]$ we find $I_2 = \{9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21\}$. Mode profiles are shown in Figure 26.



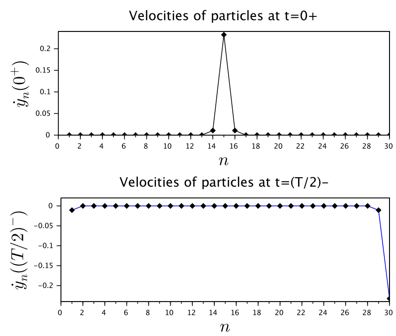
(a) Positions for $T = 0.997T_1$



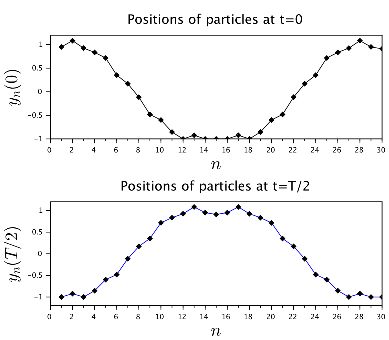
(b) Velocities for $T = 0.997T_1$



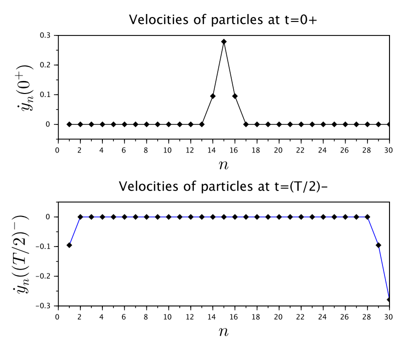
(c) Positions for $T = 0.99T_1$



(d) Velocities for $T = 0.99T_1$

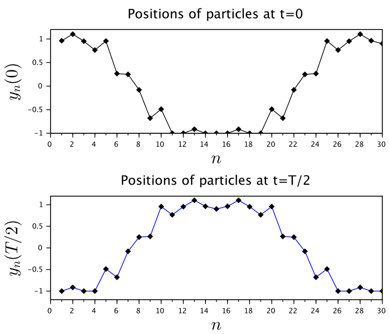


(e) Positions for $T = 0.9825924T_1$

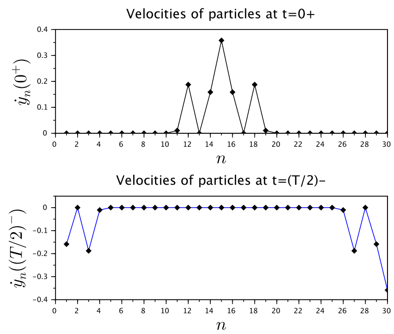


(f) Velocities for $T = 0.9825924T_1$

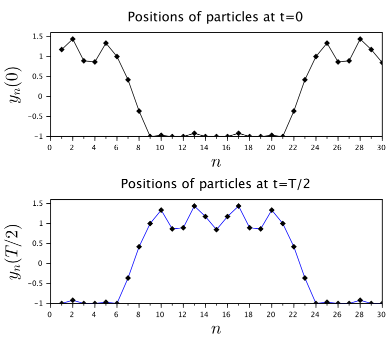
Fig. 26: A second branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi\,(1+4\gamma\sin^2(\pi/30))^{-1/2}$.



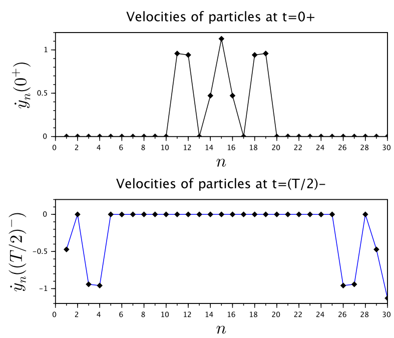
(g) Positions for $T = 0.964T_1$



(h) Velocities for $T = 0.964T_1$

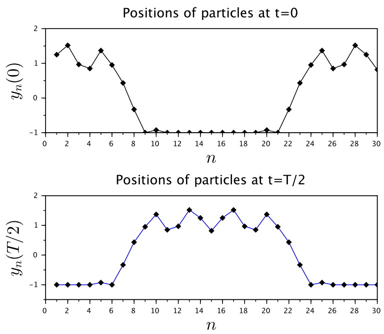


(i) Positions for $T = 0.849T_1$

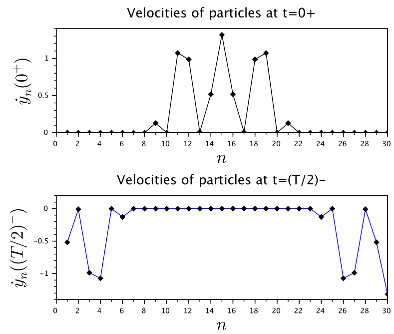


(j) Velocities for $T = 0.849T_1$

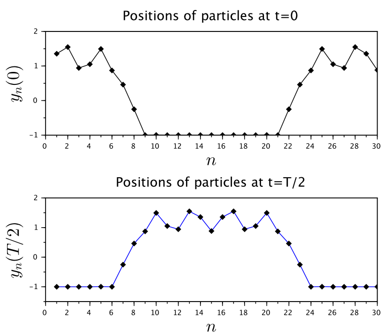
Fig. 26: A second branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi\left(1+4\gamma\sin^2(\pi/30)\right)^{-1/2}$ (continued).



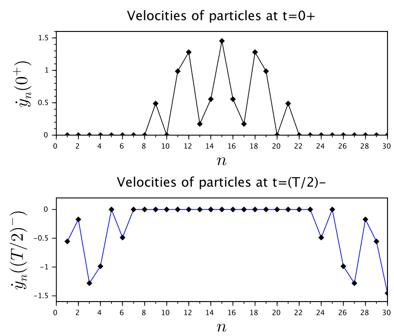
(k) Positions for $T = 0.835T_1$



(l) Velocities for $T = 0.835T_1$



(m) Positions for $T = 0.81T_1$



(n) Velocities for $T = 0.81T_1$

Fig. 26: A second branch of long-wavelength normal modes for $\gamma = 1$ and $T_1 = 2\pi\left(1+4\gamma\sin^2(\pi/30)\right)^{-1/2}$ (continued).

4 Discussion

In this work, we have studied the existence and stability of nonsmooth modes (either spatially localized or extended) in a chain of coupled impact oscillators, for rigid impacts without energy dissipation. We have obtained analytical solutions with an arbitrary number of impacting particles at small coupling, and have computed such solutions numerically for larger coupling constants. Different solution branches corresponding to stable or unstable breathers, multibreathers and nonsmooth normal modes have been found.

The computation of periodic solutions based on the above approach is much more effective than numerical continuation of periodic solutions based on stiff compliant models. In the latter case, impacts are described by smooth nonlinear Hertzian type potentials leading to stiff ODE and costly numerical continuation.

Several extensions of this work could be considered. It would be interesting to perform the continuation of periodic solutions while allowing switches in the mode patterns. In addition, the study of more complex types of nonsmooth modes would be of great interest. In particular, one could allow particles to realize several impacts per period [38] or display sticking phases after a grazing contact [28]. The inclusion of dissipative impacts and forcing and the application of the method to more complex finite-element models of continuous impacting systems constitute additional challenging directions.

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